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# From the Laplacian with variable magnetic field to the electric Laplacian in the semiclassical limit

Nicolas Raymond

May 5, 2012

## Abstract

We consider a twisted magnetic Laplacian with Neumann condition on a smooth and bounded domain of  $\mathbb{R}^2$  in the semiclassical limit  $h \rightarrow 0$ . Under generic assumptions, we prove that the eigenvalues admit a complete asymptotic expansion in powers of  $h^{1/4}$ .

## 1 Introduction and main results

Let  $\Omega$  be an open bounded and simply connected subset of  $\mathbb{R}^2$  with smooth boundary. Let us consider a smooth vector potential  $\mathbf{A}$  such that  $\beta = \nabla \times \mathbf{A} > 0$  on  $\overline{\Omega}$  and  $a$  a smooth and positive function on  $\overline{\Omega}$ . We are interested in estimating the eigenvalues  $\lambda_n(h)$  of the operator  $P_{h,\mathbf{A}} = (ih\nabla + \mathbf{A})a(ih\nabla + \mathbf{A})$  whose domain is given by:

$$\text{Dom}(P_{h,\mathbf{A}}) = \left\{ \psi \in L^2(\Omega) : (-ih\nabla + \mathbf{A})a(-ih\nabla + \mathbf{A})\psi \in L^2(\Omega) \right. \\ \left. \text{and } (-ih\nabla + \mathbf{A})\psi \cdot \nu = 0 \text{ on } \partial\Omega \right\}.$$

The corresponding quadratic form denoted by  $Q_{h,\mathbf{A}}$  is defined on  $H^1(\Omega)$  by:

$$Q_{h,\mathbf{A}}(\psi) = \int_{\Omega} a(x) |(-ih\nabla + \mathbf{A})\psi|^2 dx.$$

By gauge invariance, this is standard that the spectrum of  $P_{h,\mathbf{A}}$  depends on the magnetic field  $\beta = \nabla \times \mathbf{A}$ , but not on the potential  $\mathbf{A}$  itself.

### 1.1 Motivation and presentation of the problem

**Motivation and context** Before stating our main result, we should briefly describe the context and the motivations of this paper. As much in 2D as in 3D, the magnetic Laplacian, corresponding to the case when  $a = 1$ , appears in the theory of superconductivity when studying the third critical field  $H_{C_3}$  that appears after the linearization of the Ginzburg-Landau functional (see for instance [21, 22] and also the

book of Fournais and Helffer [14]). It turns out that  $H_{C_3}$  can be related to the lowest eigenvalue of the magnetic Laplacian in the regime  $h \rightarrow 0$ .

In fact, the case which is mainly investigated in the literature is the case when the magnetic field is constant. In 2D, the two terms asymptotics is done in the case of the disk by Bauman, Phillips and Tang in [4] (see also [5] and [10]) and is generalized by Helffer and Morame (see [18]) to smooth and bounded domains. The asymptotic expansion at any order of all the lowest eigenvalues is proved by Fournais and Helffer in [13]. In 3D, one can mention the celebrated paper [19] giving the two terms asymptotics of the first eigenvalue.

When the magnetic field is variable (and  $a = 1$ ), less results are known. In 2D, the paper of Lu and Pan [21] provides a one term asymptotics of the lowest eigenvalue and [25] gives the two term asymptotics under generic assumptions (we can also mention [17] dealing with the case without boundary and which provides a full asymptotic expansion of the eigenvalues). In 3D, for the one term asymptotics, one can mention [22] and for a three terms asymptotics upper bound [26] (see also [27] where a complete asymptotics is proved for a toy model).

Here we consider a twist factor  $a > 0$ . As we will see, the presence of  $a$  (which is maybe not the main point of this paper) will not complicate the philosophy of the analysis even if it will lead to use generalizations of the Feynman-Hellmann theorems (such generalizations were introduced by physicists to analyze the anisotropic Ginzburg-Landau functional, see [11]). In fact, this additional term obliges to have a more synthetic sight of the structure of the magnetic Laplacian. The motivation to add this term comes from [7] where the authors deal with the anisotropic Ginzburg-Landau functional (which is an effective mass model). We can also refer to [3] where closely related problems appear. Moreover, we will see that the quantity to minimize to get the lowest energy is the function  $a\beta$  so that this situation recalls what happens in 3D in [22, 26] and where the three terms asymptotics is still not established.

Under generic assumptions, we will prove in this paper that the eigenvalues  $\lambda_n(h)$  admit complete asymptotic expansions in powers of  $h^{1/4}$ .

**Heuristics** Let us discuss a little bit the heuristics to understand the problem. Let us fix a point  $x_0 \in \overline{\Omega}$ . If  $x_0 \in \Omega$  and if we approximate the vector potential  $\mathbf{A}$  by its linear part, we can locally write the magnetic Laplacian as:

$$a(x_0)(h^2 D_x^2 + (hD_y - \beta(x_0)x)^2) + \text{lower order terms.}$$

The lowest eigenvalue can be computed after a Fourier transform with respect to  $y$  and a translation with respect to  $x$  (which reduces to a 1D harmonic oscillator) ; it provides an eigenvalue  $a(x_0)\beta(x_0)h$ . If  $x_0 \in \partial\Omega$  and considering the standard boundary coordinates  $(s, t)$  ( $t > 0$  being the distance to the boundary and  $s$  the curvilinear coordinate), we get the approximation:

$$h^2 D_t^2 + (hD_s - \beta(x_0)t)^2 + \text{lower order terms.}$$

The shape of this formal approximation invites us to recall basic properties of the de Gennes operator.

**The de Gennes operator** For  $\xi \in \mathbb{R}$ , we consider the Neumann realization  $H_\xi$  in  $L^2(\mathbb{R}_+)$  associated with the operator

$$-\frac{d^2}{dt^2} + (t - \xi)^2, \quad \text{Dom}(H_\xi) = \{u \in B^2(\mathbb{R}_+) : u'(0) = 0\}. \quad (1.1)$$

One knows (see [9]) that it has compact resolvent and its lowest eigenvalue is denoted  $\mu(\xi)$ ; the associated  $L^2$ -normalized and positive eigenstate is denoted by  $u_\xi = u(\cdot, \xi)$  and is in the Schwartz class. The function  $\xi \mapsto \mu(\xi)$  admits a unique minimum in  $\xi = \xi_0$  and we let:

$$\Theta_0 = \mu(\xi_0), \quad (1.2)$$

$$C_1 = \frac{u_{\xi_0}^2(0)}{3}. \quad (1.3)$$

Let us also recall identities established by [5]. For  $k \in \mathbb{N}$ , we let:

$$M_k = \int_{t>0} (t - \xi_0)^k |u_{\xi_0}(t)|^2 dt$$

and we have:

$$M_0 = 1, \quad M_1 = 0, \quad M_2 = \frac{\Theta_0}{2}, \quad M_3 = \frac{C_1}{2} \quad \text{and} \quad \frac{\mu''(\xi_0)}{2} = 3C_1 \sqrt{\Theta_0}. \quad (1.4)$$

## 1.2 Main result

Let us introduce the general assumptions under which we will work all along this paper. As already mentioned, the natural invariant associated with the operator is the function  $a\beta$ . We will assume that:

$$\Theta_0 \min_{\partial\Omega} a(x)\beta(x) < \min_{\Omega} a(x)\beta(x) \quad (1.5)$$

and that

$$x \in \partial\Omega \mapsto a(x)\beta(x) \text{ admits a unique and non degenerate minimum at } x_0. \quad (1.6)$$

**Remark 1.1** Assumption 1.5 is automatically satisfied when the magnetic field is constant (it is sometimes called surface superconductivity condition) and Assumption 1.6 excludes the case of the constant magnetic field. Therefore our generic assumption deals with a complementary situation analyzed in [13], that is the situation with a generically variable magnetic field.

Let us state our first rough estimate of the  $n$ -th eigenvalue  $\lambda_n(h)$  of  $P_{h,\mathbf{A}}$  that we will prove in this paper:

**Proposition 1.2** Under Assumptions (1.5) and (1.6), for all  $n \geq 1$ , we have:

$$\lambda_n(h) = \Theta_0 h a(x_0)\beta(x_0) + O(h^{5/4}).$$

From this proposition, we see that the asymptotics of  $\lambda_n(h)$  is related to local properties of  $P_{h,\Lambda}$  near the point of the boundary  $x_0$ . That is why we are led to introduce the standard system of local coordinates  $(s, t)$  near  $x_0$ , where  $t$  is the distance to the boundary and  $s$  the curvilinear coordinate on the boundary (see (2.1)). We denote by  $\Phi : (s, t) \mapsto x$  the corresponding local diffeomorphism. We write the Taylor expansions:

$$\tilde{a}(s, t) = a(\Phi(s, t)) = 1 + a_1 s + a_2 t + a_{11} s^2 + a_{12} s t + a_{22} t^2 + O(|s|^3 + |t|^3) \quad (1.7)$$

and

$$\tilde{\beta}(s, t) = \beta(\Phi(s, t)) = 1 + b_1 s + b_2 t + b_{11} s^2 + b_{12} s t + b_{22} t^2 + O(|s|^3 + |t|^3), \quad (1.8)$$

where we have assumed the normalization:

$$a(x_0) = \beta(x_0) = 1. \quad (1.9)$$

Let us translate the generic assumptions (1.5) and (1.6). The critical point condition becomes:

$$a_1 = -b_1 \quad (1.10)$$

and the non-degeneracy property reformulates:

$$b_{11} + a_1 b_1 + a_{11} = a_{11} + b_{11} - a_1^2 = \alpha > 0. \quad (1.11)$$

We can now state the main result of this paper:

**Theorem 1.3** *We assume Assumptions (1.5), (1.6) and the normalization condition (1.9). For all  $n \geq 1$ , there exist a sequence  $(\gamma_{n,j})_{j \geq 0}$  and  $h_0 > 0$  such that for all  $h \in (0, h_0)$ , we have:*

$$\lambda_n(h) \underset{h \rightarrow 0}{\sim} h \sum_{j \geq 0} \gamma_{n,j} h^{\frac{j}{4}}.$$

Moreover, we have, for all  $n \geq 1$ :

$$\begin{aligned} \gamma_{n,0} &= \Theta_0, \quad \gamma_{n,1} = 0, \\ \gamma_{n,2} &= C(k_0, a_2, b_2) + (2n-1) \left( \frac{\alpha \Theta_0 \mu''(\xi_0)}{2} \right)^{1/2}, \end{aligned}$$

with:

$$C(k_0, a_2, b_2) = -C_1 k_0 + \frac{3C_1}{2} a_2 + \left( \frac{C_1}{2} + \xi_0 \Theta_0 \right) b_2.$$

### 1.3 Comments around the main theorem

Let us first notice that Theorem 1.3 completes the one of Fournais and Helffer [13, Theorem 1.1] dealing with a constant magnetic field (see also [13, Remark 1.2] where the variable magnetic field case is left as an open problem).

It turns out that Theorem 1.3 generalizes [25, Theorem 1.7]. Moreover, as a consequence of the asymptotics of the eigenvalues (which are simple for  $h$  small enough),

we also get the corresponding asymptotics for the eigenfunctions. These eigenfunctions are approximated (in the  $L^2$  sense) by the power series which we will use as quasimodes (see (2.10)). In particular the eigenfunctions are approximated by functions in the form:

$$u_{\xi_0}(h^{-1/2}t)g(h^{-1/4}s),$$

where  $g$  is a renormalized Hermite function.

As we will see in the proof, the construction of appropriate trial functions can give a hint of the natural scales of the problem ( $h^{1/2}$  with respect to  $t$  and  $h^{1/4}$  with respect to  $s$ ). Nevertheless, as far as we know, there are no structural explanations in the literature of the double scales phenomena related to the magnetic Laplacian.

In this paper we will enlighten the fact that, thanks to conjugations of the magnetic Laplacian (by explicit unitary transforms in the spirit of Egorov theorem, see [12]), we can reduce the study to an electric Laplacian which is in the Born-Oppenheimer form (see [23]). The main point in the Born-Oppenheimer approximation is that it naturally involves two different scales (related to the so-called slow and fast variables).

As we recalled at the beginning of the introduction many papers deal with the two or three first terms of  $\lambda_1(h)$  and do not analyze  $\lambda_n(h)$  (for  $n \geq 2$ ), see for instance [19, 25]. One could think that it is just a technical extension. But, as it can be seen in [13], the difficulty of the extension relies on the microlocalization properties of the operator: The authors have to combine a very fine analysis using pseudo-differential calculus (to catch the a priori behavior of the eigenfunctions with respect to a phase variable) and the Grushin reduction machinery (see [15]). Let us emphasize that these microlocalization properties are one of the deepest features of the magnetic Laplacian and are often living in the core of the proofs (see for instance [19, Sections 11.2 and 13.2] and [13, Sections 5 and 6]). We will see in this paper how we can avoid the introduction of the pseudo-differential (or abstract functional) calculus. In fact we will also avoid the Grushin formalism by keeping only the main idea behind it: We can use the true eigenfunctions as quasimodes for the first order approximation of  $P_{h,A}$  and deduce a tensorial structure of the eigenfunctions.

In our investigation we will introduce successive changes of variables and unitary transforms such as changes of gauge and weighted Fourier transforms (which are all associated with canonical transformations of the symbol). By doing this we will reduce the symbol of the operator (or equivalently reduce the quadratic form) thanks to the a priori localization estimates. By gathering all these transforms one would obtain a Fourier integral operator which transforms (modulo lower order terms) the magnetic Laplacian into an electric Laplacian in the Born-Oppenheimer form. For this normal form we can prove Agmon estimates with respect to a phase variable. These estimates involve, for the normal form, strong microlocalization estimates and spare us, for instance, the multiple commutator estimates needed in [13, Section 5].

## 1.4 Scheme of the proof

Let us now describe the scheme of the proof. In Section 2, we perform a construction of quasimodes and quasi-eigenvalues thanks to a formal expansion in power series of the operator. This analysis relies on generalizations of the Feynman-Hellmann

formula and of the Virial theorem which were already introduced in [26] and which are an alternative to the Grushin approach used in [13]. Then, we use the spectral theorem to infer the existence of spectrum near each constructed power series. In Section 3, we prove a rough lower bound for the lowest eigenvalues and deduce Agmon estimates with respect to the variable  $t$  which provide a localization of the lowest eigenfunctions in a neighborhood of the boundary of size  $h^{1/2}$ . In Section 4, we improve the lower bound of Section 3 and deduce a localization of size  $h^{1/4}$  with respect to the tangential coordinate  $s$ . In Section 5, we prove a lower bound for  $Q_{h,\mathbf{A}}$  thanks to the definition of "magnetic coordinates" and we reduce the study to a model operator (in the Born-Oppenheimer form) for which we are able to estimate the spectral gap between the lowest eigenvalues.

## 2 Accurate construction of quasimodes

This section is devoted to the proof of the following theorem:

**Theorem 2.1** *For all  $n \geq 1$ , there exists a sequence  $(\gamma_{n,j})_{j \geq 0}$  such that, for all  $J \geq 0$ , there exist  $h_0 > 0$ ,  $C > 0$  such that:*

$$d \left( h \sum_{j=0}^J \gamma_{n,j} h^{j/4}, \sigma(P_{h,\mathbf{A}}) \right) \leq C h^{\frac{J+1}{4}}.$$

Moreover, we have, for all  $n \geq 1$ :

$$\begin{aligned} \gamma_{n,0} &= \Theta_0, \quad \gamma_{n,1} = 0, \\ \gamma_{n,2} &= C(k_0, a_2, b_2) + (2n-1) \left( \frac{(a_{11} + b_{11} - a_1^2) \Theta_0 \mu''(\xi_0)}{2} \right)^{1/2}. \end{aligned}$$

The proof of Theorem 2.1 is based on a construction of quasimodes for  $P_{h,\mathbf{A}}$  localized near  $x_0$ .

**Local coordinates  $(s, t)$**  We use the local coordinates  $(s, t)$  near  $x_0 = (0, 0)$ , where  $t(x) = d(x, \partial\Omega)$  and  $s(x)$  is the tangential coordinate of  $x$ . We choose a parametrization of the boundary:

$$\gamma : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \rightarrow \partial\Omega.$$

Let  $\nu(s)$  be the unit vector normal to the boundary, pointing inward at the point  $\gamma(s)$ . We choose the orientation of the parametrization  $\gamma$  to be counter-clockwise, so that:

$$\det(\gamma'(s), \nu(s)) = 1.$$

The curvature  $k(s)$  at the point  $\gamma(s)$  is given in this parametrization by:

$$\gamma''(s) = k(s)\nu(s).$$

The map  $\Phi$  defined by:

$$\begin{aligned}\Phi : \mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times ]0, t_0[ &\rightarrow \Omega \\ (s, t) &\mapsto \gamma(s) + t\nu(s),\end{aligned}\tag{2.1}$$

is clearly a diffeomorphism, when  $t_0$  is sufficiently small, with image

$$\Phi(\mathbb{R}/(|\partial\Omega|\mathbb{Z}) \times ]0, t_0[) = \{x \in \Omega \mid d(x, \partial\Omega) < t_0\} = \Omega_{t_0}.$$

We let:

$$\begin{aligned}\tilde{A}_1(s, t) &= (1 - tk(s))\mathbf{A}(\Phi(s, t)) \cdot \gamma'(s), \quad \tilde{A}_2(s, t) = \mathbf{A}(\Phi(s, t)) \cdot \nu(s), \\ \tilde{\beta}(s, t) &= \beta(\Phi(s, t)),\end{aligned}$$

and we get:

$$\partial_s \tilde{A}_2 - \partial_t \tilde{A}_1 = (1 - tk(s))\tilde{\beta}(s, t).$$

The quadratic form becomes:

$$Q_{h,\mathbf{A}}(\psi) = \int \tilde{a}(1 - tk(s)) |(-ih\partial_t + \tilde{A}_2)\psi|^2 + \tilde{a}(1 - tk(s))^{-1} |(-ih\partial_s + \tilde{A}_1)\psi|^2 ds dt.$$

In a (simply connected) neighborhood of  $(0, 0)$ , we can choose a gauge such that:

$$\tilde{A}_1(s, t) = - \int_{t_1}^t (1 - t'k(s)) \tilde{\beta}(s, t') dt', \quad \tilde{A}_2 = 0.\tag{2.2}$$

**The operator in the coordinates  $(s, t)$**  Near  $x_0$  and using a suitable gauge (see (2.2)), we are led to construct quasimodes for the following the operator:

$$\begin{aligned}\mathcal{L}(s, -ih\partial_s; t, -ih\partial_t) &= -h^2(1 - tk(s))^{-1} \partial_t (1 - tk(s)) \tilde{a} \partial_t \\ &\quad + (1 - tk(s))^{-1} (-ih\partial_s + \tilde{A})(1 - tk(s))^{-1} \tilde{a} (-ih\partial_s + \tilde{A}),\end{aligned}$$

where (see (1.8)):

$$\tilde{A}(s, t) = (t - \xi_0 h^{1/2}) + b_1 s (t - \xi_0 h^{1/2}) + (b_2 - k_0) \frac{t^2}{2} + b_{11} s^2 (t - \xi_0 h^{1/2}) + O(|t|^3 + |st^2|).$$

Let us now perform the scaling:

$$s = h^{1/4} \sigma \text{ and } t = h^{1/2} \tau.$$

The operator becomes:

$$\mathcal{L}(h) = \mathcal{L}(h^{1/4} \sigma, -ih^{3/4} \partial_\sigma; h^{1/2} \tau, -ih^{1/2} \partial_\tau).$$

We can formally write  $\mathcal{L}(h)$  as a power series:

$$\mathcal{L}(h) \sim h \sum_{j \geq 0} \mathcal{L}_j h^{j/4},$$



where

$$\mathcal{L}_0 = -\partial_\tau^2 + (\tau - \xi_0)^2, \quad (2.3)$$

$$\begin{aligned} \mathcal{L}_1 &= -a_1\sigma\partial_\tau^2 - 2i\partial_\sigma(\tau - \xi_0) + a_1(\tau - \xi_0)^2\sigma + 2b_1\sigma\tau(\tau - \xi_0) \\ &= a_1\sigma H_{\xi_0} - 2i\partial_\sigma(\tau - \xi_0) + 2b_1\sigma(\tau - \xi_0)^2, \end{aligned} \quad (2.4)$$

$$\begin{aligned} \mathcal{L}_2 &= -a_2\tau\partial_\tau^2 - a_2\partial_\tau + k_0\partial_\tau + 2k_0\tau(\tau - \xi_0)^2 + a_2\tau(\tau - \xi_0)^2 \\ &\quad + (b_2 - k_0)\tau^2(\tau - \xi_0) - ia_1(\tau - \xi_0) \\ &\quad + \sigma^2(a_{11}H_{\xi_0} - a_1^2(\tau - \xi_0)^2 + 2b_{11}(\tau - \xi_0)^2) \\ &\quad - \partial_\sigma^2 - 2ia_1(\tau - \xi_0)\sigma\partial_\sigma + ia_1(\tau - \xi_0)\partial_\sigma\sigma. \end{aligned} \quad (2.5)$$

The aim is now to define good quasimodes for  $\mathcal{L}(h)$ . Before starting the construction, we shall recall in the next subsection a few formulas coming from perturbation theory.

## 2.1 Feynman-Hellmann and Virial formulas

For  $\rho > 0$  and  $\xi \in \mathbb{R}$ , let us introduce the Neumann realization on  $\mathbb{R}_+$  of:

$$H_{\rho,\xi} = -\rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2.$$

By scaling, we observe that  $H_{\rho,\xi}$  is unitarily equivalent to  $H_\xi$  and that  $H_{1,\xi} = H_\xi$  (the corresponding eigenfunction is  $u_{1,\xi} = u_\xi$ ). The form domain of  $H_{\rho,\xi}$  is  $B^1(\mathbb{R}_+)$  and is independent from  $\rho$  and  $\xi$  so that the family  $(H_{\rho,\xi})_{\rho>0, \xi\in\mathbb{R}}$  is an holomorphic family of type (B) (see [20, p. 395]). The lowest eigenvalue of  $H_{\rho,\xi}$  is  $\mu(\xi)$  and we will denote by  $u_{\rho,\xi}$  the corresponding normalized eigenfunction:

$$u_{\rho,\xi}(\tau) = \rho^{1/4}u_\xi(\rho^{1/2}\tau).$$

Since  $u_\xi$  satisfies the Neumann condition, we observe that  $\partial_\rho^m \partial_\xi^n u_{\rho,\xi}$  also satisfies it. In order to lighten the notation and when it is not ambiguous we will write  $H$  for  $H_{\rho,\xi}$ ,  $u$  for  $u_{\rho,\xi}$  and  $\mu$  for  $\mu(\xi)$ .

The main idea is now to take derivatives of:

$$Hu = \mu u \quad (2.6)$$

with respect to  $\rho$  and  $\xi$ . Taking the derivative with respect to  $\rho$  and  $\xi$ , we get the proposition:

**Proposition 2.2** *We have:*

$$(H - \mu)\partial_\xi u = 2(\rho^{1/2}\tau - \xi)u + \mu'(\xi)u \quad (2.7)$$

and

$$(H - \mu)\partial_\rho u = -\rho^{-2}\partial_\tau^2 - \xi\rho^{-1}(\rho^{1/2}\tau - \xi) - \rho^{-1}\tau(\rho^{1/2}\tau - \xi)^2. \quad (2.8)$$

Moreover, we get:

$$(H - \mu)(Su) = Xu, \quad (2.9)$$

where

$$X = -\frac{\xi}{2}\mu'(\xi) + \rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2$$

and

$$S = -\frac{\xi}{2}\partial_\xi - \rho\partial_\rho.$$

*Proof:* Taking the derivatives with respect to  $\xi$  and  $\rho$  of (2.6), we get:

$$(H - \mu)\partial_\xi u = \mu'(\xi)u - \partial_\xi H u$$

and

$$(H - \mu)\partial_\rho u = -\partial_\rho H.$$

We have:  $\partial_\xi H = -2(\rho^{1/2}\tau - \xi)$  and  $\partial_\rho H = \rho^{-2}\partial_\rho^2 + \rho^{-1/2}\tau(\rho^{1/2}\tau - \xi)$ .  $\square$

Taking  $\rho = 1$  and  $\xi = \xi_0$  in (2.7), we deduce, with the Fredholm alternative:

**Corollary 2.3** *We have:*

$$(H_{\xi_0} - \mu(\xi_0))v_{\xi_0} = 2(t - \xi_0)u_{\xi_0},$$

with:

$$v_{\xi_0} = (\partial_\xi u_\xi)|_{\xi=\xi_0}.$$

Moreover, we have:

$$\int_{\tau>0} (\tau - \xi_0)u_{\xi_0}^2 d\sigma d\tau = 0.$$

**Corollary 2.4** *We have, for all  $\rho > 0$ :*

$$\int_{\tau>0} (\rho^{1/2}\tau - \xi_0)u_{\rho,\xi_0}^2 d\sigma d\tau = 0$$

and:

$$\int_{\tau>0} (\tau - \xi_0) (\partial_\rho u)_{\rho=1,\xi=\xi_0} u d\sigma d\tau = -\frac{\xi_0}{4}.$$

**Corollary 2.5** *We have:*

$$(H_{\xi_0} - \mu(\xi_0))S_0 u = (\partial_\tau^2 + (\tau - \xi_0)^2) u_{\xi_0},$$

where:

$$S_0 u = -(\partial_\rho u_{\rho,\xi})|_{\rho=1,\xi=\xi_0} - \frac{\xi_0}{2}v_{\xi_0}.$$

Moreover, we have:

$$\|\partial_\tau u_{\xi_0}\|^2 = \|(\tau - \xi_0)u_{\xi_0}\|^2 = \frac{\Theta_0}{2}.$$

The next three propositions deal with the second derivatives of (2.6) with respect to  $\xi$  and  $\rho$ .

**Proposition 2.6** *We have:*

$$(H_\xi - \mu(\xi))w_{\xi_0} = 4(\tau - \xi_0)v_{\xi_0} + (\mu''(\xi_0) - 2)u_{\xi_0},$$

with

$$w_{\xi_0} = (\partial_\xi^2 u_\xi)|_{\xi=\xi_0}.$$

Moreover, we have:

$$\int_{\tau>0} (\tau - \xi_0)v_{\xi_0}u_{\xi_0} d\sigma d\tau = \frac{2 - \mu''(\xi_0)}{4}.$$

*Proof:* Taking the derivative of (2.7) with respect to  $\xi$  (with  $\rho = 1$ ), we get:

$$(H_\xi - \mu(\xi))\partial_\xi^2 u_\xi = 2\mu'(\xi)\partial_\xi u_\xi + 4(\tau - \xi)\partial_\xi u_\xi + (\mu''(\xi) - 2)u_\xi.$$

It remains to take  $\xi = \xi_0$  and to write the Fredholm alternative.  $\square$

**Proposition 2.7** *We have:*

$$\begin{aligned} (H - \mu)(\partial_\rho^2 u)_{\rho=1, \xi=\xi_0} &= -2(\partial_\tau^2 + (\tau - \xi_0)^2)(\partial_\rho u)_{\rho=1, \xi=\xi_0} \\ &\quad - 2\xi_0(\tau - \xi_0)(\partial_\rho u)_{\rho=1, \xi=\xi_0} + (2\partial_\tau^2 - \frac{\xi_0\tau}{2})u_{\xi_0} \end{aligned}$$

and:

$$\langle (\partial_\tau^2 + (\tau - \xi_0)^2)(\partial_\rho u)_{\rho=1, \xi=\xi_0}, u_{\xi_0} \rangle = -\frac{\Theta_0}{2}.$$

*Proof:* We just have to take the derivative of (2.8) with respect to  $\rho$  and  $\rho = 1$ ,  $\xi = \xi_0$ . To get the second identity, we use the Fredholm alternative, Corollary 2.4 and Corollary 2.5.  $\square$

Taking the derivative of (2.9) with respect to  $\rho$ , we find:

**Lemma 2.8** *We have:*

$$\begin{aligned} (H - \mu)(\partial_\rho S u)_{\rho=1, \xi=\xi_0} &= (-\partial_\tau^2 + \tau(\tau - \xi_0))u_{\xi_0} - (\partial_\rho H)_{\rho=1, \xi=\xi_0}(S_0 u) \\ &\quad + (\partial_\tau^2 + (\tau - \xi_0)^2)(\partial_\rho u)_{\rho=1, \xi=\xi_0} \end{aligned}$$

and

$$\langle (\partial_\rho H)_{\rho=1, \xi=\xi_0}(S_0 u), u \rangle = \frac{\Theta_0}{2}.$$

**Lemma 2.9** *We have:*

$$\langle (\tau - \xi_0)S_0 u, u_{\xi_0} \rangle = \frac{\xi_0}{8}\mu''(\xi_0).$$

*Proof:* We have:

$$\mu'(\xi) = -2 \int_{\tau>0} (\rho^{1/2}\tau - \xi) u_{\rho,\xi}^2 d\sigma d\tau$$

and:

$$S_0\mu' = -2 \int_{\tau>0} S_0(\rho^{1/2}\tau - \xi) u_{\xi_0}^2 d\sigma d\tau - 4 \int_{\tau>0} (\tau - \xi_0) S_0 u u_{\xi_0} d\sigma d\tau.$$

□

Combining Lemmas 2.8 and 2.9, we deduce:

**Proposition 2.10** *We have:*

$$\langle (-\partial_\tau^2 - (\tau - \xi_0)^2) S_0 u, u_{\xi_0} \rangle = -\frac{\Theta_0}{2} + \frac{\Theta_0}{8} \mu''(\xi_0).$$

**Proposition 2.11** *We have:*

$$\langle (\partial_\tau^2 + (\tau - \xi_0)^2) v_{\xi_0}, u_{\xi_0} \rangle = \frac{\xi_0 \mu''(\xi_0)}{4}.$$

*Proof:* We take the derivative of (2.7) with respect to  $\rho$  (after having fixed  $\xi = \xi_0$ ):

$$(H - \mu) (\partial_\xi u)_{\xi=\xi_0} = 2(\rho^{1/2}\tau - \xi_0) u_{\rho,\xi_0}.$$

We deduce:

$$(H - \mu) (\partial_\rho \partial_\xi u)_{\rho=1, \xi=\xi_0} = -(\partial_\rho H)_{\rho=1, \xi=\xi_0} v_{\xi_0} + \tau u_{\xi_0} + 2(\tau - \xi_0) (\partial_\rho u)_{\rho=1, \xi=\xi_0}.$$

The Fredholm alternative provides:

$$\langle (\partial_\tau^2 + \tau(\tau - \xi_0)) v_{\xi_0}, u_{\xi_0} \rangle = \xi_0 + 2 \langle (\tau - \xi_0) (\partial_\rho u)_{\rho=1, \xi=\xi_0}, u_{\xi_0} \rangle = \frac{\xi_0}{2},$$

where we have used Corollary 2.4. □

We have now the elements to perform an accurate construction of quasimodes.

## 2.2 Construction

We look for quasimodes expressed as power series:

$$\psi \sim \sum_{j \geq 0} \psi_j h^{j/4}$$

and eigenvalues:

$$\lambda \sim h \sum_{j \geq 0} \lambda_j h^{j/4}$$

so that, in the sense of formal series:

$$\mathcal{L}(h)\psi \sim \lambda\psi.$$

**Term in  $h$**  We consider the equation:

$$(\mathcal{L}_0 - \lambda_0)\psi_0 = 0.$$

We are led to take  $\lambda_0 = \Theta_0$  and  $\psi_0(\sigma, \tau) = f_0(\sigma)u_{\xi_0}(\tau)$ .

**Term in  $h^{5/4}$**  We want to solve the equation:

$$(\mathcal{L}_0 - \Theta_0)\psi_1 = \lambda_1\psi_0 - \mathcal{L}_1\psi_0.$$

We have, by using that  $b_1 = -a_1$  and Proposition 2.2:

$$(\mathcal{L}_0 - \Theta_0)(\psi_1 - if_0'(\sigma)v_{\xi_0} - a_1\sigma f_0(\sigma)S_0u) = \lambda_1u_{\xi_0}.$$

This implies that  $\lambda_1 = 0$  and we take:

$$\psi_1(\sigma, \tau) = if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u + f_1(\sigma)u_{\xi_0}(\tau),$$

$f_0$  and  $f_1$  being to determine.

**Term in  $h^{3/2}$**  We consider the equation:

$$(\mathcal{L}_0 - \Theta_0)\psi_2 = \lambda_2\psi_0 - \mathcal{L}_1\psi_1 - \mathcal{L}_2\psi_0.$$

Let us rewrite this equation by using the expression of  $\psi_1$ :

$$(\mathcal{L}_0 - \Theta_0)\psi_2 = \lambda_2\psi_0 - \mathcal{L}_1(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u) - \mathcal{L}_1(f_1(\sigma)u_{\xi_0}) - \mathcal{L}_2\psi_0.$$

With Proposition 2.2, we deduce:

$$\begin{aligned} & (\mathcal{L}_0 - \Theta_0)(\psi_2 - if_1'(\sigma)v_{\xi_0} - a_1\sigma f_1(\sigma)S_0u) \\ &= \lambda_2\psi_0 - \mathcal{L}_1(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u) - \mathcal{L}_2\psi_0. \end{aligned}$$

We take the partial scalar product (with respect to  $\tau$ ) of the r.h.s. with  $u_{\xi_0}$  and we get the equation:

$$\langle \mathcal{L}_1(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u) + \mathcal{L}_2\psi_0, u_{\xi_0} \rangle_\tau = \lambda_2f_0.$$

This equation can be written in the form:

$$(AD_\sigma^2 + B_1\sigma D_\sigma + B_2D_\sigma\sigma + C\sigma^2 + D)f_0 = \lambda_2f_0.$$

**Terms in  $D_\sigma^2$**  Let us first analyze  $\langle \mathcal{L}_2u_{\xi_0}, u_{\xi_0} \rangle$ . This is easy to see that this terms is 1. Let us then analyze  $\langle \mathcal{L}_1\psi_1, u_{\xi_0} \rangle$ . With Proposition 2.6, we deduce that this term is  $-2\langle (\tau - \xi_0)v_{\xi_0}u_{\xi_0} \rangle = \frac{\mu''(\xi_0)}{2} - 1$ . We get:  $A = \frac{\mu''(\xi_0)}{2} > 0$ .

**Terms in  $\sigma^2$**  Let us collect the terms of  $\langle \mathcal{L}_2 u_{\xi_0}, u_{\xi_0} \rangle$ . We get:

$$\Theta_0 a_{11} + 2b_{11} \langle (\tau - \xi_0)^2 u_{\xi_0}, u_{\xi_0} \rangle - a_1^2 \langle (\tau - \xi_0)^2 u_{\xi_0}, u_{\xi_0} \rangle.$$

With Corollary 2.5, this term is equal to:

$$\Theta_0 a_{11} + \Theta_0 b_{11} - \frac{\Theta_0}{2} a_1^2.$$

Let us analyze the terms coming from  $\langle \mathcal{L}_1 \psi_1, u_{\xi_0} \rangle$ . We obtain the term:

$$a_1^2 \langle (-\partial_\tau^2 - (\tau - \xi_0)^2) S_0 u, u_{\xi_0} \rangle = -\frac{\Theta_0}{2} a_1^2 + \Theta_0 \frac{\mu''(\xi_0)}{8} a_1^2,$$

where we have used Proposition 2.10. Thus, we have:

$$C = \Theta_0 a_{11} + \Theta_0 b_{11} - \Theta_0 a_1^2 + \frac{\Theta_0}{8} \mu''(\xi_0) a_1^2 > 0.$$

**Terms in  $\sigma D_\sigma$**  This term only comes from  $\langle \mathcal{L}_1 \psi_1, u_{\xi_0} \rangle$ . It is equal to:

$$a_1 \langle (\partial_\tau^2 + (\tau - \xi_0)^2) v_{\xi_0}, u_{\xi_0} \rangle = a_1 \frac{\xi_0 \mu''(\xi_0)}{4},$$

where we have used Proposition 2.11.

**Terms in  $D_\sigma \sigma$**  This term is:

$$2a_1 \langle (\tau - \xi_0) S_0 u, u_{\xi_0} \rangle = a_1 \frac{\xi_0 \mu''(\xi_0)}{4},$$

where we have applied Lemma 2.9.

**Value of  $D$**  We have:

$$D = \langle (-a_2 \tau \partial_\tau^2 - a_2 \partial_\tau + k_0 \partial_\tau + 2k_0 \tau (\tau - \xi_0)^2 + a_2 \tau (\tau - \xi_0)^2) u_{\xi_0}, u_{\xi_0} \rangle \\ + \langle ((b_2 - k_0) \tau^2 (\tau - \xi_0) - i a_1 (\tau - \xi_0)) u_{\xi_0}, u_{\xi_0} \rangle.$$

Using the relations (1.4) and the definition of  $C_1$  given in (1.3), we get:

$$D = C(k_0, a_2, b_2).$$

Let us introduce the quadratic form which is fundamental in the analyzis. We let:

$$\mathcal{Q}(\sigma, \eta) = \frac{\mu''(\xi_0)}{2} \eta^2 + a_1 \frac{\xi_0 \mu''(\xi_0)}{4} \eta \sigma + a_1 \frac{\xi_0 \mu''(\xi_0)}{4} \sigma \eta \\ + \Theta_0 \left( a_{11} + b_{11} - a_1^2 + a_1^2 \frac{\mu''(\xi_0)}{8} \right) \sigma^2.$$

**Lemma 2.12**  $\mathcal{Q}$  is definite and positive.

*Proof:* We notice that  $\mu''(\xi_0) > 0$  and  $a_{11} + b_{11} - a_1^2 + a_1^2 \frac{\mu''(\xi_0)}{8} > 0$ . The determinant is given by:

$$\begin{aligned} & \Theta_0 \frac{\mu''(\xi_0)}{2} \left( a_{11} + b_{11} - a_1^2 + a_1^2 \frac{\mu''(\xi_0)}{8} \right) - a_1^2 \frac{\Theta_0 \mu''(\xi_0)^2}{16} \\ &= \frac{\Theta_0 \mu''(\xi_0)}{2} (a_{11} + b_{11} - a_1^2) > 0. \end{aligned}$$

□

We immediately deduce that  $\mathcal{Q}(\sigma, -i\partial_\sigma)$  is unitarily equivalent to an harmonic oscillator and that the increasing sequence of its eigenvalues is given by

$$\left\{ (2n+1) \left( \frac{\Theta_0 \mu''(\xi_0)}{2} (a_{11} + b_{11} - a_1^2) \right)^{1/2} \right\}_{n \in \mathbb{N}}.$$

The compatibility equation becomes:

$$\mathcal{Q}(\sigma, D_\sigma) f_0 = (\lambda_2 - D) f_0.$$

Thus, we choose  $\lambda_2$  such that  $\lambda_2 - D$  is in the spectrum of  $\mathcal{Q}(\sigma, D_\sigma)$  and we take for  $f_0$  the corresponding normalized eigenfunction (which is in the Schwartz class). For that choice of  $f_0$ , we can consider the unique solution  $\psi_2^\perp$  (which is in the Schwartz class) of:

$$(\mathcal{L}_0 - \Theta_0) \psi_2^\perp = \lambda_2 \psi_0 - \mathcal{L}_1 (i f_0'(\sigma) v_{\xi_0} + a_1 \sigma f_0(\sigma) S_0 u) - \mathcal{L}_2 \psi_0$$

satisfying  $\langle \psi_2^\perp, u_{\xi_0} \rangle = 0$ . It follows that  $\psi_2$  is in the form:

$$\psi_2 = \psi_2^\perp(\sigma, \tau) + i f_1'(\sigma) v_{\xi_0} + a_1 \sigma f_1(\sigma) S_0 u + f_2(\sigma) u_{\xi_0},$$

where  $f_1$  and  $f_2$  are still to be determined.

**Higher order terms** Let  $N \geq 2$ . Let us assume that, for  $0 \leq j \leq N-2$ , the functions  $\psi_j$  are determined and belong to the Schwartz class. Moreover, let us also assume that, for  $j = N-1, N$ , we can write:

$$\psi_j(\sigma, \tau) = \psi_j^\perp(\sigma, \tau) + i f_{j-1}'(\sigma) v_{\xi_0} + a_1 \sigma f_{j-1}(\sigma) S_0 u + f_j(\sigma) u_{\xi_0},$$

where the  $(\psi_j^\perp)_{j=N-1, N}$  and  $f_{N-2}$  are determined functions in the Schwartz class and where the  $(f_j)_{j=N-1, N}$  are not determined. Finally, we also assume that the  $(\lambda_j)_{0 \leq j \leq N}$  are determined. We notice that this recursion assumption is satisfied for  $N = 2$ . Let us write the equation of order  $N+1$ :

$$\begin{aligned} (\mathcal{L}_0 - \Theta_0) \psi_{N+1} &= \lambda_{N+1} \psi_0 - \mathcal{L}_1 \psi_N + (\lambda_2 - \mathcal{L}_2) \psi_{N-1} \\ &\quad - \mathcal{L}_{N+1} \psi_0 + \sum_{j=1}^{N-2} (\lambda_{N+1-j} - \mathcal{L}_{N+1-j}) \psi_j. \end{aligned}$$

This equation takes the form:

$$(\mathcal{L}_0 - \Theta_0)\psi_{N+1} = \lambda_{N+1}\psi_0 - \mathcal{L}_1\psi_N + (\lambda_2 - \mathcal{L}_2)\psi_{N-1} + F_N(\sigma, \tau),$$

where  $F_N$  is a determined function in the Schwartz class by recursion assumption. Using Proposition 2.2, we can rewrite:

$$\begin{aligned} & (\mathcal{L}_0 - \Theta_0)(\psi_{N+1} - if'_N(\sigma)v_{\xi_0} - a_1\sigma f_N(\sigma)S_0u) \\ &= \lambda_{N+1}\psi_0 - \mathcal{L}_1\left(\psi_N^\perp(\sigma, \tau) + if'_{N-1}(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0u\right) + (\lambda_2 - \mathcal{L}_2)\psi_{N-1} \\ & \quad + F_N(\sigma, \tau) \\ &= \lambda_{N+1}\psi_0 - \mathcal{L}_1\left(if'_{N-1}(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0u\right) + (\lambda_2 - \mathcal{L}_2)(f_{N-1}u_{\xi_0}) \\ & \quad + G_N(\sigma, \tau), \end{aligned}$$

where  $G_N$  is a determined function of the Schwartz class. We now write the Fredholm condition. The same computation as previously leads to an equation in the form:

$$\mathcal{Q}(\sigma, -i\partial_\sigma)f_{N-1} = (\lambda_2 - C(a_2, b_2, k_0))f_{N-1} + \lambda_{N+1}f_0 + g_N(\sigma),$$

with  $g_N = \langle G_N, u_{\xi_0} \rangle_\tau$ . This can be rewritten as:

$$(\mathcal{Q}(\sigma, -i\partial_\sigma) - (\lambda_2 - C(a_2, b_2, k_0)))f_{N-1} = g_N(\sigma) + \lambda_{N+1}f_0.$$

The Fredholm condition applied to this equation provides:  $\lambda_{N+1} = -\langle g_N, f_0 \rangle_\sigma$  and a unique solution  $f_{N-1}$  in the Schwartz class such that  $\langle f_{N-1}, f_0 \rangle_\sigma = 0$ . For this choice of  $f_{N-1}$  and  $\lambda_{N+1}$ , we can consider the unique solution  $\psi_{N+1}^\perp$  (in the Schwartz class) such that:

$$\begin{aligned} & (\mathcal{L}_0 - \Theta_0)\psi_{N+1}^\perp \\ &= \lambda_{N+1}\psi_0 - \mathcal{L}_1\left(\psi_N^\perp(\sigma, \tau) + if'_{N-1}(\sigma)v_{\xi_0} + a_1\sigma f_{N-1}(\sigma)S_0u\right) + (\lambda_2 - \mathcal{L}_2)\psi_{N-1} \\ & \quad + F_N(\sigma, \tau). \end{aligned}$$

This leads to take:

$$\psi_{N+1} = \psi_{N+1}^\perp + if'_N(\sigma)v_{\xi_0} + a_1\sigma f_N(\sigma)S_0u + f_{N+1}u_{\xi_0}.$$

This ends the proof of the recursion. Thus, we have constructed two sequences  $(\lambda_j)_j$  and  $(\psi_j)_j$  which depend on  $n$  (through the choice of  $f_0$ ). Let us write  $\lambda_{n,j}$  for  $\lambda_j$  and  $\psi_{n,j}$  for  $\psi_j$  to emphasize this dependence.

**Conclusion: proof of Theorem 2.1** Let us consider a smooth cutoff function  $\chi_0$  near  $x_0$ . For  $n \geq 1$  and  $J \geq 0$ , we let:

$$\psi_h^{[n,J]}(x) = \chi_0(x) \sum_{j=0}^J \psi_{n,j}(h^{-1/4}s(x), h^{-1/2}t(x))h^{j/4} \quad (2.10)$$



and:

$$\lambda_h^{[n,J]} = \sum_{j=0}^J \lambda_{n,j} h^{j/4}.$$

Using the fact that the  $\psi_j$  are in the Schwartz class, we get:

$$\| (P_{h,\mathbf{A}} - \lambda_h^{[n,J]}) \psi_h^{[n,J]} \| \leq C(n, J) h^{\frac{J+1}{4}} \| \psi_h^{[n,J]} \|.$$

Thanks to the spectral theorem, we deduce Theorem 2.1.

### 3 Rough lower bound and consequence

This section is devoted to establish a rough lower bound for  $\lambda_n(h)$ . In particular, we give the first term of the asymptotics and deduce the so-called normal Agmon estimates which are rather standard (see for instance [18, 13, 25]).

#### 3.1 A first lower bound

We now aim at proving a lower bound:

**Proposition 3.1** *We have:*

$$\lambda_n(h) \geq \Theta_0 h a(x_0) \beta(x_0) - C h^{5/4}.$$

*Proof:* We use a partition of unity with balls  $D_j$  of size  $h^\rho$  and satisfying:

$$\sum_j \chi_j^2 = 1 \text{ and } \sum_j \|\nabla \chi_j\|^2 \leq C h^{-2\rho}.$$

The so-called IMS formula (cf. [8]) provides:

$$Q_{h,\mathbf{A}}(\psi) = \sum_j Q_{h,\mathbf{A}}(\chi_j \psi) - h^2 \sum_j \int_{\Omega} a \|\nabla \chi_j\|^2 |\psi|^2 dx$$

and thus:

$$Q_{h,\mathbf{A}}(\psi) \geq \sum_j Q_{h,\mathbf{A}}(\chi_j \psi) - C h^{2-2\rho} \|\psi\|^2.$$

In each ball, we approximate  $a$  by a constant:

$$Q_{h,\mathbf{A}}(\chi_j \psi) \geq (a(x_j) - C h^\rho) \|(-ih\nabla + \mathbf{A})(\chi_j \psi)\|^2.$$

If  $D_j$  does not intersect the boundary, then:

$$\|(-ih\nabla + \mathbf{A})(\chi_j \psi)\|^2 \geq h \int_{\Omega} \beta(x) |\chi_j \psi|^2 dx.$$

We deduce:

$$Q_{h,\mathbf{A}}(\chi_j \psi) \geq (a(x_j) \beta(x_j) h - C h^{1+\rho}) \|\chi_j \psi\|^2.$$

If  $D_j$  intersects the boundary, we can assume that its center is on the boundary and we write in the local coordinates (up to a change of gauge):

$$Q_{h,A}(\chi_j \psi) \geq (1 - Ch^\rho) \int \tilde{a} \left( h^2 |\partial_t(\chi_j \psi)|^2 + |(-ih\partial_s + \tilde{A}_1)(\chi_j \psi)|^2 \right) ds dt.$$

We deduce:

$$Q_{h,A}(\chi_j \psi) \geq (1 - Ch^\rho)(a(x_j) - Ch^\rho) \int h^2 |\partial_t(\chi_j \psi)|^2 + |(-ih\partial_s + \tilde{A}_1)(\chi_j \psi)|^2 ds dt.$$

We approximate  $A_1$  by its linear approximation  $A_1^{\text{lin}}$  and we have:

$$\begin{aligned} & \int h^2 |\partial_t(\chi_j \psi)|^2 + |(-ih\partial_s + \tilde{A}_1)(\chi_j \psi)|^2 ds dt \\ & \geq (1 - \varepsilon) \int h^2 |\partial_t(\chi_j \psi)|^2 + |(-ih\partial_s + \tilde{A}_1^{\text{lin}})(\chi_j \psi)|^2 ds dt - C\varepsilon^{-1} \int |x - x_j|^4 |\chi_j \psi|^2 dx \\ & \geq ((1 - \varepsilon)\Theta_0\beta(x_j)h - C\varepsilon^{-1}h^{4\rho}) \|\chi_j \psi\|^2. \end{aligned}$$

To optimize the remainder, we choose:  $\varepsilon = h^{2\rho-1/2}$ . Then, we take  $\rho = \frac{3}{8}$  and the conclusion follows.  $\square$

### 3.2 Normal Agmon estimates: localization in $t$

We now prove the following (weighted) localization estimates:

**Proposition 3.2** *Let us consider a smooth cutoff function  $\chi$  supported in a fixed neighborhood of the boundary. Let  $(\lambda_n(h), \psi_h)$  be an eigenpair of  $P_{h,A}$ . For all  $\delta \geq 0$ , there exist  $\varepsilon_0, C \geq 0$  and  $h_0$  such that, for  $h \in (0, h_0)$ :*

$$\begin{aligned} & \|e^{\varepsilon_0 t(x)h^{-1/2} + \delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2 \leq C \|e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2, \\ & Q_{h,A} \left( e^{\varepsilon_0 t(x)h^{-1/2} + \delta \chi(x)|s(x)|h^{-1/4}} \psi_h \right) \leq Ch \|e^{\delta \chi(x)|s(x)|h^{-1/4}} \psi_h\|^2. \end{aligned}$$

*Proof:* The proof is based on a technique of Agmon (see for instance [1, 2, 16]). Let us recall the IMS formula ; we have, for an eigenpair  $(\lambda_n(h), \psi_h)$ :

$$Q_{h,A} (e^\Phi \psi_h) = \lambda_n(h) \|e^\Phi \psi_h\|^2 + h^2 \|a^{1/2} \nabla \Phi e^\Phi \psi_h\|^2.$$

We take:

$$\Phi = \varepsilon_0 t(x)h^{-1/2} + \delta \chi(x)|s(x)|h^{-1/4},$$

where  $\chi$  is a smooth cutoff function supported near the boundary. We use a partition of unity  $\chi_j$  with balls of size  $Rh^{1/2}$  with  $R$  large enough and we get:

$$\sum_j \left( Q_{h,A} (\chi_j e^\Phi \psi_h) - \lambda_n(h) \|\chi_j e^\Phi \psi_h\|^2 - CR^{-2}h - h^2 \|\chi_j a^{1/2} \nabla \Phi e^\Phi \psi_h\|^2 \right) \leq 0.$$

We now distinguish the balls intersecting the boundary and the others. For the interior balls, we have the lower bound, for  $\eta > 0$  and  $h$  small enough:

$$Q_{h,\mathbf{A}}(\chi_j e^\Phi \psi_h) \geq \left( a(x_j) \beta(x_j) h - C h^{3/2} \right) \|\chi_j e^\Phi \psi_h\|^2.$$

For the boundary balls, we have:

$$Q_{h,\mathbf{A}}(\chi_j e^\Phi \psi_h) \geq \left( \Theta_0 a(x_j) \beta(x_j) h - C h^{3/2} \right) \|\chi_j e^\Phi \psi_h\|^2.$$

Let us now split the sum:

$$\begin{aligned} & \sum_{j \text{ int}} \int \left( a(x_j) \beta(x_j) h - \Theta_0 a(x_0) \beta(x_0) h - C h^{3/2} - C R^{-2} h - C h^2 \|\nabla \Phi\|^2 \right) |\chi_j e^\Phi \psi_h|^2 dx \\ & \leq C h \sum_{j \text{ bnd}} \|\chi_j e^\Phi \psi_h\|^2. \end{aligned}$$

We can notice that:

$$\|\nabla \Phi\|^2 \leq C(\varepsilon_0^2 h^{-1} + \delta^2 h^{-1/2}).$$

Taking  $R$  large enough,  $\varepsilon_0$  and  $h$  small enough and using (1.6), we get the existence of  $c > 0$  such that:

$$a(x_j) \beta(x_j) h - \Theta_0 a(x_0) \beta(x_0) h - C h^{3/2} - C R^{-2} h - C h^2 \|\nabla \Phi\|^2 \geq c h.$$

We deduce:

$$c \sum_{j \text{ int}} \|\chi_j e^\Phi \psi_h\|^2 \leq C \sum_{j \text{ bnd}} \|\chi_j e^\Phi \psi_h\|^2.$$

Due to support considerations, we can write:

$$C \sum_{j \text{ bnd}} \|\chi_j e^\Phi \psi_h\|^2 \leq \tilde{C} \sum_{j \text{ bnd}} \|\chi_j e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2.$$

Thus, we infer:

$$\|e^\Phi \psi_h\|^2 \leq \tilde{C} \|e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2.$$

We deduce that:

$$\sum_j Q_{h,\mathbf{A}}(\chi_j e^\Phi \psi_h) \leq C h \|e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2.$$

and thus:

$$Q_{h,\mathbf{A}}(e^\Phi \psi_h) \leq C h \|e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2.$$

□

**Corollary 3.3** *Let  $\eta \in (0, \frac{1}{2}]$ . Let  $(\lambda_n(h), \psi_h)$  be an eigenpair of  $P_{h,\mathbf{A}}$ . For all  $\delta \geq 0$ , there exist  $\varepsilon_0, C \geq 0$  and  $h_0$  such that, for  $h \in (0, h_0)$ :*

$$\begin{aligned} & \|\chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2 \leq C \|\chi_{h,\eta} e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2, \\ & Q_{h,\mathbf{A}} \left( \chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta \chi(x) |s(x)| h^{-1/4}} \psi_h \right) \leq C h \|\chi_{h,\eta} e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2, \end{aligned}$$

where  $\chi_{h,\eta}(x) = \hat{\chi}(t(x) h^{-1/2+\eta})$  and with  $\hat{\chi}$  a smooth cutoff function being 1 near 0.

*Proof:* With Proposition 3.2, we have:

$$\|\chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2 \leq C \|e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2.$$

We can write:

$$\|e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2 = \|\chi_{h,\eta} e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2 + \|\sqrt{1 - \chi_{h,\eta}} e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2.$$

Using Proposition 3.2, we have the estimate:

$$\begin{aligned} & \|\sqrt{1 - \chi_{h,\eta}} e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2 \\ &= \|\sqrt{1 - \chi_{h,\eta}} e^{-\varepsilon_0 t(x) h^{-1/2}} e^{\chi(x) \varepsilon_0 t(x) h^{-1/2} + \delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2 \\ &= O(h^\infty) \|e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2. \end{aligned}$$

The IMS formula provides:

$$Q_{h,\mathbf{A}} (e^\Phi \psi_h) = Q_{h,\mathbf{A}} (\chi_{h,\eta} e^\Phi \psi_h) + Q_{h,\mathbf{A}} (\sqrt{1 - \chi_{h,\eta}} e^\Phi \psi_h) + O(h^{1+2\eta}) \|e^\Phi \psi_h\|^2.$$

□

**Corollary 3.4** *Let  $\eta \in (0, \frac{1}{2}]$ . Let  $(\lambda_n(h), \psi_h)$  be an eigenpair of  $P_{h,\mathbf{A}}$ . For all  $\delta \geq 0$ , there exist  $\varepsilon_0, C \geq 0$  and  $h_0$  such that, for  $h \in (0, h_0)$ :*

$$\begin{aligned} \|\chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta \chi(x) |s(x)| h^{-1/4}} (-ih\partial_s + \tilde{A}_1) \psi_h\|^2 &\leq Ch \|\chi_{h,\eta} e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2, \\ \|\chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta \chi(x) |s(x)| h^{-1/4}} (-ih\partial_t + \tilde{A}_2) \psi_h\|^2 &\leq Ch \|\chi_{h,\eta} e^{\delta \chi(x) |s(x)| h^{-1/4}} \psi_h\|^2. \end{aligned}$$

## 4 Order of the second term: localization in $s$

It is well-known that the order of the second term in the asymptotics of  $\lambda_n(h)$  is closely related to localization properties of the corresponding eigenfunctions. The aim of this section is to establish such properties. Let us mention that similar estimates were proved in [25] through a technical analysis. Here we give a less technical proof using a very rough functional calculus.

**Proposition 4.1** *Under the generic assumptions, there exist  $C > 0$  and  $h_0 > 0$  such that for  $h \in (0, h_0)$ :*

$$\lambda_n(h) \geq \Theta_0 a(x_0) \beta(x_0) h - Ch^{3/2}.$$

Moreover, for all  $\delta \geq 0$ , there exist  $C > 0$  and  $h_0 > 0$  such that for  $h \in (0, h_0)$ :

$$\int e^{2\delta \chi(x) |s| h^{-1/4}} |\psi|^2 ds dt \leq C \|\psi\|^2.$$

*Proof:* Let us recall the so-called IMS formula (see for instance [8]) ; we have, for an eigenpair  $(\lambda_n(h), \psi)$ :

$$Q_{h,\mathbf{A}}(e^\Phi \psi) - \lambda_n(h) \|e^\Phi \psi\|^2 - h^2 \|a^{1/2} \nabla \Phi e^\Phi \psi\|^2 = 0.$$

We take:

$$\Phi = \delta \chi(x) |s(x)| h^{-1/4}, \quad \text{with } \delta \geq 0.$$

The idea is now to prove a suitable lower bound for  $Q_{h,\mathbf{A}}$ . We use a partition of unity with balls of size  $h^{1/4}$ . We get the lower bound:

$$Q_{h,\mathbf{A}}(e^\Phi \psi) \geq \sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2} \|e^\Phi \psi\|^2,$$

where

$$\psi_j = \chi_{j,h} e^\Phi \psi$$

and we deduce:

$$\sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2} \|\psi_j\|^2 - \lambda_n(h) \|\psi_j\|^2 \leq 0, \quad (4.1)$$

since  $\|\nabla \Phi\| \leq Ch^{-1/2}$ .

**Interior balls** Considering the balls intersecting the boundary, we get:

$$\sum_{j \text{ int}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ int}} (a(x_j) \beta(x_j) h - Ch^{5/4}) \|\psi_j\|^2. \quad (4.2)$$

**Boundary balls** Let us consider the  $j$  such that  $D_j$  intersects the boundary. Using first the normal Agmon estimates, we have the lower bound:

$$\sum_{j \text{ bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ bnd}} \int \tilde{a}(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) ds dt - Ch^{3/2} \|\psi\|^2,$$

where we have used the IMS formula to get:

$$\begin{aligned} & \sum_{j \text{ bnd}} \int t \tilde{a}(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) ds dt \\ & \leq C \int_{0 < t < t_0} t \tilde{a}(|(-ih\partial_t + \tilde{A}_2)e^\Phi \psi|^2 + |(ih\partial_s + \tilde{A}_1)e^\Phi \psi|^2) ds dt + Ch^{3/2} \|e^\Phi \psi\|^2. \end{aligned}$$

Using again the normal estimates and also the size of the balls, we get:

$$\sum_{j \text{ bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ bnd}} \int \tilde{a}_j^{\text{lin}}(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) ds dt - Ch^{3/2} \|\psi\|^2, \quad (4.3)$$

where

$$\tilde{a}_j^{\text{lin}} = a_j + (s - s_j) \partial_s \tilde{a}(x_j).$$

**Case  $|s_j| \geq s_0$**  Let us consider the boundary balls such that  $|s_j| \geq s_0$ . Using the size of the balls, we get the lower bound:

$$\begin{aligned} & \int \tilde{a}_j^{\text{lin}} (|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) dsdt \\ & \geq (\Theta_0 a(x_j) \beta(x_j) h - Ch^{5/4}) \|\psi_j\|^2 \geq \Theta_0 (1 + \varepsilon) a(x_0) \beta(x_0) h \|\psi_j\|^2. \end{aligned} \quad (4.4)$$

**Case  $|s_j| \leq s_0$**  Let us consider the boundary balls such that  $|s_j| \leq s_0$ . In each ball, we can use a new gauge so that:

$$\begin{aligned} & \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2) dsdt \\ & = \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (|h\partial_t \psi_j|^2 + |(ih\partial_s + \tilde{A}_1^{\text{new}})\psi_j|^2) dsdt, \end{aligned}$$

where  $\tilde{A}_1^{\text{new}}$  satisfies:

$$|\tilde{A}_1^{\text{new}} - t\tilde{\beta}_j^{\text{lin}}| \leq C(t|s - s_j|^2 + t^2),$$

with:

$$\tilde{\beta}_j^{\text{lin}} = \tilde{\beta}_j + \partial_s \tilde{\beta}(x_j)(s - s_j).$$

We obtain, thanks to the estimates of Agmon:

$$\begin{aligned} & \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (|h\partial_t \psi_j|^2 + |(ih\partial_s + \tilde{A}_1^{\text{new}})\psi_j|^2) dsdt \\ & \geq (1 - h^{1/2}) \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int \tilde{a}_j^{\text{lin}} (h^2 |\partial_t \psi_j|^2 + |(ih\partial_s + t\tilde{\beta}_j^{\text{lin}})\psi_j|^2) dsdt - Ch^{3/2} \|\psi\|^2. \end{aligned} \quad (4.5)$$

In each ball, we use the change of variables (which is a scaling with respect to  $\tau$  depending on  $\sigma$ ):

$$\sigma = s \quad \text{and} \quad \tau = \left( \tilde{\beta}_j^{\text{lin}} \right)^{1/2} t.$$

We can write:

$$\partial_t = \left( \tilde{\beta}_j^{\text{lin}} \right)^{1/2} \partial_\tau \quad \text{and} \quad \partial_s = \partial_\sigma + \partial_s \left( \tilde{\beta}_j^{\text{lin}} \right)^{1/2} \partial_\tau$$

and

$$dsdt = \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau.$$

We obtain:

$$\begin{aligned} & \int h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_s + t\tilde{\beta}_j^{\text{lin}})\hat{\psi}_j|^2 dsdt \\ & \geq (1 - h^{1/2}) \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left( h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \partial_\sigma + \tau) \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ & \quad - Ch^{3/2} \int |\tau \partial_\tau \hat{\psi}_j|^2 d\sigma d\tau. \end{aligned} \quad (4.6)$$

With the normal Agmon estimates, we have:

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \int |\tau \partial_\tau \hat{\psi}_j|^2 d\sigma d\tau \leq C \|\psi\|^2.$$

We can notice that the Dirichlet realization on  $(-\tilde{s}_0, \tilde{s}_0)$  of  $D_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2}$  is self-adjoint on  $L^2 \left( \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma \right)$ . Thus, we shall commute  $D_\sigma$  and  $\left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2}$  and control the remainder due to the commutator. We can write:

$$\begin{aligned} & \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left( h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \partial_\sigma + \tau) \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ &= \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} h^2 |\partial_t \hat{\psi}_j|^2 \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ &+ \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} |(ih \partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} - ih \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} + \tau) \hat{\psi}_j|^2 \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau. \end{aligned}$$

We can estimate the double product:

$$\begin{aligned} & 2h\Re \left( \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left( ih \partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau \right) \hat{\psi}_j i \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} \overline{\hat{\psi}_j} \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \right) \\ &= -2h^2 \Re \left( \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} \partial_\sigma \left( \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \hat{\psi}_j \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \overline{\hat{\psi}_j} d\sigma d\tau \right) \\ &= -h^2 \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \partial_\sigma \left\{ \tilde{\beta}_j^{\text{lin}} \right\}^{-1/2} \partial_\sigma \left( \left| \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \hat{\psi}_j \right|^2 \right) d\sigma d\tau = O(h^2) \|\psi_j\|^2, \end{aligned}$$

where we have used an integration by parts for the last estimate. We deduce:

$$\begin{aligned} & \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left( h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \partial_\sigma + \tau) \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \quad (4.7) \\ & \geq \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left( h^2 |\partial_t \hat{\psi}_j|^2 + |(ih \partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ & - Ch^2 \|\psi_j\|^2. \end{aligned}$$

For  $s_0$  small enough, we have, using the non-degeneracy, for  $s$  such that  $|s| \leq \tilde{s}_0$  (with  $\tilde{s}_0$  slightly bigger than  $s_0$ ):

$$\tilde{a}_j^{\text{lin}}(s) \tilde{\beta}_j^{\text{lin}}(s) \geq a(x_0) \beta(x_0) + \frac{\alpha}{4} |s|^2.$$

Let us analyze the integral:

$$\begin{aligned} & \int |\sigma (ih \partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \hat{\psi}_j|^2 \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\ &= \int |(ih \partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \sigma \hat{\psi}_j - ih \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \hat{\psi}_j|^2 \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau. \end{aligned}$$

We must estimate the double product:

$$\begin{aligned}
& 2\Re \int \left( (ih\partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \sigma \hat{\psi}_j ih \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \overline{\hat{\psi}_j} \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\
&= -2h^2 \Re \int \left( \partial_\sigma \left( \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \sigma \hat{\psi}_j \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \overline{\hat{\psi}_j} \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\
&= -h^2 \int \partial_\sigma \left| \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} \hat{\psi}_j \right|^2 \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau + O(h^2) \|\hat{\psi}_j\|^2 \\
&= O(h^2) \|\hat{\psi}_j\|^2.
\end{aligned}$$

We infer:

$$\begin{aligned}
& \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left( h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\
&\geq a(x_0) \beta(x_0) \int \left( h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\
&+ \frac{\alpha}{4} \int \left( h^2 |\partial_t (\sigma \hat{\psi}_j)|^2 + |(ih\partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau) \sigma \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\
&- Ch^2 \|\hat{\psi}_j\|^2.
\end{aligned}$$

We recall that, for all  $\xi \in \mathbb{R}$ :

$$\int \left( h^2 |\partial_t \phi|^2 + |(\tau - h\xi - \xi_0 h^{1/2}) \phi|^2 \right) d\tau \geq h\mu(\xi_0 + h^{1/2}\xi) \|\phi\|^2 \geq \Theta_0 \|\phi\|^2.$$

We infer with the functional calculus:

$$\begin{aligned}
& \int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \left( h^2 |\partial_t \hat{\psi}_j|^2 + |(ih\partial_\sigma \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} + \tau - \xi_0 h^{1/2}) \hat{\psi}_j|^2 \right) \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau \\
&\geq h\Theta_0 \int \left( a(x_0) \beta(x_0) + \frac{\alpha}{4} \sigma^2 \right) |\hat{\psi}_j|^2 \left( \tilde{\beta}_j^{\text{lin}} \right)^{-1/2} d\sigma d\tau - Ch^2 \|\hat{\psi}_j\|^2.
\end{aligned} \tag{4.8}$$

**Lower bound for  $\lambda_n(h)$**  If we take  $\delta = 0$ , we deduce, with (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8):

$$\lambda_n(h) \|\psi\|^2 \geq \sum_j \Theta_0 h a(x_0) \beta(x_0) \int |\psi_j|^2 dx - Ch^{3/2} \|\psi\|^2.$$

**Tangential Agmon estimate** Gathering all the estimates, we deduce the existence of  $c > 0$  such that:

$$\begin{aligned}
& \sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \left( \Theta_0 h \int \left( a(x_0) \beta(x_0) + \frac{\alpha}{4} s^2 \right) |\psi_j|^2 ds dt - \Theta_0 h \|\psi_j\|^2 - Ch^{3/2} \|\psi_j\|^2 \right) \\
&+ \sum_{\substack{j \text{ bnd} \\ |s_j| \geq s_0}} ch \|\psi_j\|^2 + \sum_{j \text{ int}} ch \|\psi_j\|^2 \leq 0
\end{aligned}$$



and:

$$\sum_{\substack{j \text{ bnd} \\ 2C_0 h^{1/4} \leq |s_j| \leq s_0}} \left( \Theta_0 h \int \frac{\alpha}{4} s^2 |\psi_j|^2 ds dt - Ch^{3/2} \|\psi_j\|^2 \right) \leq Ch^{3/2} \|\psi_0\|^2 \leq Ch^{3/2} \|\psi\|^2.$$

Taking  $C_0$  large enough, we infer:

$$\sum_{\substack{j \text{ bnd} \\ 2C_0 h^{1/4} \leq |s_j| \leq s_0}} \|\psi_j\|^2 \leq C \|\psi\|^2$$

so that:

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \|\psi_j\|^2 \leq C \|\psi\|^2$$

and:

$$\sum_j \|\psi_j\|^2 = \|e^\Phi \psi\|^2 \leq C \|\psi\|^2.$$

□

Let us write an immediate corollary (see Corollaries 3.3 and 3.4).

**Corollary 4.2** *Let  $(\eta_1, \eta_2) \in (0, \frac{1}{2}] \times (0, \frac{1}{4}]$ . Let  $(\lambda_n(h), \psi_h)$  be an eigenpair of  $P_{h,A}$ . For all  $(k, l) \in \mathbb{N}$ , there exist  $C \geq 0$  and  $h_0 > 0$  such that, for  $h \in (0, h_0)$ :*

$$\begin{aligned} \|\chi_{h,\eta_1,\eta_2} s^k t^l \psi_h\|^2 &\leq Ch^{k/2} h^l \|\psi_h\|^2, \\ \|\chi_{h,\eta_1,\eta_2} s^k t^l (-ih\partial_s + \tilde{A}_1) \psi_h\|^2 &\leq Chh^{k/2} h^l \|\psi_h\|^2, \\ \|\chi_{h,\eta_1,\eta_2} s^k t^l (-ih\partial_t + \tilde{A}_2) \psi_h\|^2 &\leq Chh^{k/2} h^l \|\psi_h\|^2. \end{aligned}$$

where  $\chi_{h,\eta_1,\eta_2}(x) = \hat{\chi}(t(x)h^{-1/2+\eta_1})\hat{\chi}(s(x)h^{-1/4+\eta_2})$ . Moreover, we have:

$$\begin{aligned} \|(1 - \chi_{h,\eta_1,\eta_2}) s^k t^l \psi_h\|^2 &= O(h^\infty) \|\psi_h\|^2, \\ \|(1 - \chi_{h,\eta_1,\eta_2}) s^k t^l (-ih\partial_s + \tilde{A}_1) \psi_h\|^2 &= O(h^\infty) \|\psi_h\|^2, \\ \|(1 - \chi_{h,\eta_1,\eta_2}) s^k t^l (-ih\partial_t + \tilde{A}_2) \psi_h\|^2 &= O(h^\infty) \|\psi_h\|^2. \end{aligned}$$

**Remark 4.3** In the following, each reference to the "estimates of Agmon" will be a reference to this last corollary. Moreover, at some point, the localization ideas behind Section 3 and 4, which are summarized in the last corollary, follows from the general philosophy developed in the last decade (a improvement of the approximation of the eigenvalues provides an improvement of localization and conversely). In the next section, we will strongly use these a priori estimates.

## 5 Unitary transforms and Born-Oppenheimer approximation

We use a cutoff function  $\chi_h$  near  $x_0$  with support of order  $h^{1/4-\tilde{\eta}}$  with  $\tilde{\eta} > 0$ . For all  $N \geq 1$ , let us consider  $L^2$ -normalized eigenpairs  $(\lambda_n(h), \psi_{n,h})_{1 \leq n \leq N}$  such that  $\langle \psi_{n,h}, \psi_{m,h} \rangle = 0$  when  $n \neq m$ . We consider the  $N$  dimensional space defined by:

$$\mathfrak{E}_N(h) = \text{span}_{1 \leq n \leq N} \tilde{\psi}_{n,h}, \quad \text{where} \quad \tilde{\psi}_{n,h} = \chi_h \psi_{n,h}.$$

**Remark 5.1** The estimates of Agmon of Corollary 4.2 are satisfied by all the elements of  $\mathfrak{E}_N(h)$ .

We can notice that, with the estimates of Agmon, for all  $\tilde{\psi} \in \mathfrak{E}_N(h)$ :

$$Q_{h,A}(\tilde{\psi}) \leq \lambda_N(h) \|\tilde{\psi}\|^2 + O(h^\infty) \|\tilde{\psi}\|^2. \quad (5.1)$$

In this subsection, we provide a lower bound for  $Q_{h,A}$  on  $\mathfrak{E}_N(h)$ .

**Remark 5.2** Let us underline the main spirit of this section. We are going to use successive canonical transformations of the symbol of our operator (change of variable, change of gauge, weighted Fourier transform) or equivalently of the associated quadratic form. In the spirit of the Egorov theorem, all these transformations will give rise to different remainders which can be treated thanks to the a priori localizations estimates. Then, after conjugations by these successive unitary transforms, we will reduce the analysis to the one of an electric Laplacian in the Born-Oppenheimer form.

### 5.1 Choice of gauge and new coordinates: a first lower bound

On the support of  $\chi_h$ , we use a gauge such that  $\tilde{A}_2 = 0$  and

$$|\tilde{A}_1 - \tilde{A}_1^{\text{app}}| \leq C(t^3 + |s|t^2 + |s|^2t),$$

where:

$$\tilde{A}_1^{\text{app}} = t(1 + b_1s + b_{11}s^2) + \frac{\hat{b}_2}{2}t^2 = t\hat{b}(s) - \xi_0\hat{b}(s)^{1/2}h^{1/2} + \frac{\hat{b}_2}{2}t^2,$$

where  $\hat{b}_2 = b_2 - k_0$ . We also let:

$$\tilde{a}^{\text{app}}(s, t) = 1 + a_1s + a_{11}s^2 + a_2t = \hat{a}(s) + a_2t.$$

Moreover, in this neighborhood of  $(0, 0)$ , we introduce new coordinates:

$$\tau = t(\hat{b}(s))^{1/2}, \quad \sigma = s. \quad (5.2)$$

In particular, we get:

$$\partial_t = (\hat{b}(\sigma))^{1/2} \partial_\tau, \quad \partial_s = \partial_\sigma + \frac{1}{2} \hat{b}^{-1} \partial_s \hat{b} \tau \partial_\tau$$

and:

$$dsdt = \hat{b}^{-1/2} d\sigma d\tau.$$

To simplify the notation, we let:  $p = \hat{b}^{-1/2}$ . We will also use the change of variable:

$$\check{\sigma} = \int_0^\sigma \frac{1}{p(u)} du = f(\sigma)$$

so that  $L^2(p d\sigma)$  becomes  $L^2(\check{p}^2 d\check{\sigma})$ .

This subsection is devoted to the proof of the following lower bound of  $Q_{h,A}$  on  $\mathfrak{E}_N(h)$ .

**Proposition 5.3** *There exist  $h_0 > 0$  and  $C > 0$  such that for  $h \in (0, h_0)$  and all  $\tilde{\psi} \in \mathfrak{E}_N(h)$ :*

$$Q_{h,A}(\tilde{\psi}) \geq \check{Q}_{h,\text{app}}(\check{\psi}) - Ch^{3/2+1/4} \|\tilde{\psi}\|^2, \quad (5.3)$$

where:

$$\begin{aligned} \check{Q}_{h,\text{app}}(\check{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ h\alpha\Theta_0 \int \check{\sigma}^2 |\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau, \end{aligned}$$

where  $\check{\psi}$  denotes  $\tilde{\psi}$  in the coordinates  $(\check{\sigma}, \tau)$ .

In order to prove Proposition 5.3, we will need a first lemma:

**Lemma 5.4** *There exist  $h_0 > 0$  and  $C > 0$  s. t. for  $h \in (0, h_0)$  and all  $\tilde{\psi} \in \mathfrak{E}_N(h)$ :*

$$Q_{h,A}(\tilde{\psi}) \geq \hat{Q}_{h,\text{app}}(\hat{\psi}) - Ch^{3/2+1/4} \|\tilde{\psi}\|^2.$$

where:

$$\begin{aligned} \hat{Q}_{h,\text{app}}(\hat{\psi}) &= \int m_2(\sigma, \tau) |h\partial_\tau \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &+ \int m_1(\sigma, \tau) |(h\Xi + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau, \end{aligned}$$

with

$$\begin{aligned} \Xi &= i\partial_\sigma \hat{b}^{-1/2}, \quad m_1(\sigma, \tau) = (1 + \alpha\sigma^2)(1 + a_2\tau)(1 - \tau k_0)^{-1}, \\ m_2(\sigma, \tau) &= (1 + \alpha\sigma^2)(1 + a_2\tau)(1 - \tau k_0) \end{aligned}$$

and where  $\hat{\psi}$  denotes  $\tilde{\psi}$  in the coordinates  $(\sigma, \tau)$ .

*Proof:* We have:

$$Q_{h,A}(\tilde{\psi}) = \int \tilde{a}(1 - tk(s)) |(-ih\partial_t + \tilde{A}_2) \tilde{\psi}|^2 + \tilde{a}(1 - tk(s))^{-1} |(ih\partial_s + \tilde{A}_1) \tilde{\psi}|^2 dsdt.$$

Thanks to the normal and tangential Agmon estimates, we get:

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \int \tilde{a}(1-tk_0)h^2|\partial_t\tilde{\psi}|^2 + \tilde{a}(1-tk_0)^{-1}|(ih\partial_s + \tilde{A}_1)\tilde{\psi}|^2 dsdt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

The Agmon estimates imply:

$$Q_{h,\mathbf{A}}(\tilde{\psi}) \geq \int \tilde{a}^{\text{app}}(1-tk_0)h^2|\partial_t\tilde{\psi}|^2 + \tilde{a}^{\text{app}}(1-tk_0)^{-1}|(ih\partial_s + \tilde{A}_1^{\text{app}})\tilde{\psi}|^2 dsdt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

We get:

$$\begin{aligned} Q_{h,\mathbf{A}}(\tilde{\psi}) &\geq \\ &\int \hat{a}(1+a_2t) \left( (1-tk_0)h^2|\partial_t\tilde{\psi}|^2 + (1-tk_0)^{-1} \left| (ih\partial_s + \tilde{A}_1^{\text{app}})\tilde{\psi} \right|^2 \right) dsdt \\ &\quad - Ch^{3/2+1/4}\|\tilde{\psi}\|^2. \end{aligned}$$

With the coordinates  $(\sigma, \tau)$ , we obtain:

$$\begin{aligned} &\int \hat{a}(1+a_2t)(1-tk_0)h^2|\partial_t\tilde{\psi}|^2 + (1+a_2t)(1-tk_0)^{-1}|(ih\partial_s + \tilde{A}_1^{\text{app}})\tilde{\psi}|^2 dsdt \\ &\geq \hat{Q}_h(\hat{\psi}) - Ch^{3/2+1/4}\|\tilde{\psi}\|^2, \end{aligned}$$

where:

$$\begin{aligned} \hat{Q}_h(\hat{\psi}) &= \int \tilde{m}_2(\sigma, \tau) |h\partial_\tau\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &\quad + \int \tilde{m}_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 \hat{b}^{-1/2} - h\frac{\partial_\sigma \hat{b}}{2\hat{b}^{3/2}}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau, \end{aligned}$$

where:

$$\tilde{m}_1(\sigma, \tau) = \hat{a}\hat{b}(1+a_2\tau)(1-\tau k_0)^{-1}, \quad \tilde{m}_2(\sigma, \tau) = \hat{a}\hat{b}(1+a_2\tau)(1-\tau k_0).$$

With the estimates of Agmon, we can simplify the quadratic form modulo lower order terms:

$$\begin{aligned} \hat{Q}_h(\hat{\psi}) &\geq \int \tilde{m}_2(\sigma, \tau) |h\partial_\tau\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &\quad + \int \tilde{m}_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &\quad - Ch^{3/2+1/4}\|\tilde{\psi}\|^2. \end{aligned}$$

We recall that  $\hat{a}\hat{b} = 1 + \alpha\sigma^2 + O(|\sigma|^3)$  so that, with the estimates of Agmon, we infer:

$$\begin{aligned} \hat{Q}_h(\hat{\psi}) &\geq \int m_2(\sigma, \tau) |h\partial_\tau\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &\quad + \int m_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &\quad - Ch^{3/2+1/4}\|\tilde{\psi}\|^2. \end{aligned}$$

We now want to replace  $\hat{b}^{-1/2}i\partial_\sigma$  by  $i\partial_\sigma\hat{b}^{-1/2}$  which is self-adjoint on  $L^2(\hat{b}^{-1/2}d\sigma d\tau)$ .

Writing a commutator, we get:

$$\begin{aligned} & \int m_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &= \int m_1(\sigma, \tau) |(hi\partial_\sigma\hat{b}^{-1/2} - ih(\partial_\sigma\hat{b}^{-1/2}) + \tau - \xi_0 h^{1/2} + \frac{b_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau. \end{aligned}$$

Let us consider the double product:

$$\begin{aligned} & 2h\Re \left( \int m_1(\sigma, \tau) \left( hi\partial_\sigma\hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau \right) \hat{\psi} i(\partial_\sigma\hat{b}^{-1/2}) \bar{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau \right) \\ &= 2h\Re \left( \int m_1(\sigma, \tau) \left( hi\partial_\sigma\hat{b}^{-1/2} - h\frac{b_1}{2}\tau D_\tau \right) \hat{\psi} i(\partial_\sigma\hat{b}^{-1/2}) \bar{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau \right) \\ &= -2h^2\Re \int m_1(\sigma, \tau) \left( \partial_\sigma \left( \hat{b}^{-1/2} \hat{\psi} \right) (\partial_\sigma\hat{b}^{-1/2}) \bar{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau \right) + O(h^2)\|\hat{\psi}\|^2, \end{aligned}$$

where we have used the normal Agmon estimates. We deduce that:

$$\begin{aligned} & 2\Re \left( \int m_1(\sigma, \tau) \left( hi\partial_\sigma\hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau \right) \hat{\psi} i(\partial_\sigma\hat{b}^{-1/2}) \bar{\hat{\psi}} \hat{b}^{-1/2} d\sigma d\tau \right) \\ &= -h^2 \int m_1(\sigma, \tau) (\partial_\sigma\hat{b}^{-1/2}) \partial_\sigma \left| \hat{b}^{-1/2} \hat{\psi} \right|^2 d\sigma d\tau + O(h^2)\|\hat{\psi}\|^2 \\ &= O(h^2)\|\hat{\psi}\|^2. \end{aligned}$$

This implies:

$$\begin{aligned} & \int m_1(\sigma, \tau) |(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &\geq \int m_1(\sigma, \tau) |(hi\partial_\sigma\hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \hat{\psi}|^2 \hat{b}^{-1/2} d\sigma d\tau \\ &\quad - Ch^2\|\hat{\psi}\|^2. \end{aligned}$$

□

**Proof of Proposition 5.3** We use Lemma 5.4. In the coordinates  $(\check{\sigma}, \tau)$ , we have:

$$\begin{aligned} \hat{Q}_{h,\text{app}}(\hat{\psi}) &= \int m_2(f^{-1}(\check{\sigma}), \tau) |h\partial_\tau \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &\quad + \int m_1(f^{-1}(\check{\sigma}), \tau) |(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau, \end{aligned}$$

where

$$\begin{aligned} m_1(f^{-1}(\check{\sigma}), \tau) &= (1 + \alpha f^{-1}(\check{\sigma})^2)(1 + a_2\tau)(1 - \tau k_0)^{-1}, \\ m_2(f^{-1}(\check{\sigma}), \tau) &= (1 + \alpha f^{-1}(\check{\sigma})^2)(1 + a_2\tau)(1 - \tau k_0). \end{aligned}$$

We notice that:  $f^{-1}(\check{\sigma}) = \check{\sigma} + O(|\check{\sigma}|^2)$  so that, we can use the estimates of Agmon to get:

$$\begin{aligned}\hat{Q}_{h,\text{app}}(\hat{\psi}) &\geq \int m_2(\check{\sigma}, \tau) |h\partial_\tau \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int m_1(\check{\sigma}, \tau) |(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &- Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

This inequality can be rewritten as:

$$\hat{Q}_{h,\text{app}}(\hat{\psi}) \geq \check{Q}_{h,\text{app},1}(\check{\psi}) + \check{Q}_{h,\text{app},2}(\check{\psi}) - Ch^{3/2+1/4}\|\tilde{\psi}\|^2,$$

where:

$$\begin{aligned}\check{Q}_{h,\text{app},1}(\check{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau \check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau\end{aligned}$$

and:

$$\begin{aligned}\check{Q}_{h,\text{app},2}(\check{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau(\check{\sigma}\check{\psi})|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau\end{aligned}$$

**Reduction of  $\check{Q}_{h,\text{app},2}(\check{\psi})$**  By the estimates of Agmon, we have:

$$\begin{aligned}\check{Q}_{h,\text{app},2}(\check{\psi}) &\geq \int |h\partial_\tau(\check{\sigma}\check{\psi})|^2 \check{p}^2 d\check{\sigma} d\tau \\ &+ \int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

Moreover, we get:

$$\begin{aligned}&\int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau)\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &\geq \int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

Let us analyze  $\int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau$ . We have:

$$\begin{aligned}&\int |\check{\sigma}(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau \\ &= \int |(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2})\check{\sigma}\check{\psi} - ih\check{\psi}|^2 \check{p}^2 d\check{\sigma} d\tau.\end{aligned}$$

The double product is:

$$\begin{aligned} & 2\Re \left( \int (ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p} + \tau - \xi_0 h^{1/2}) \check{\sigma} \check{\psi} i h \check{\bar{\psi}} \check{p}^2 d\check{\sigma} d\tau \right) \\ &= -2h^2 \Re \left( \int (\check{p}^{-1}\partial_{\check{\sigma}}\check{p}) \check{\sigma} \check{\psi} \check{\bar{\psi}} \check{p}^2 d\check{\sigma} d\tau \right). \end{aligned}$$

But, we have:

$$2\Re \left( \int \partial_{\check{\sigma}}(\check{\sigma}\check{p}\check{\psi}) \check{\bar{p}}\check{\bar{\psi}} d\check{\sigma} d\tau \right) = 2\Re \left( \int \check{p}\check{\psi} \check{\bar{p}}\check{\bar{\psi}} d\check{\sigma} d\tau \right) + \int \check{\sigma} \partial_{\check{\sigma}} |\check{p}\check{\psi}|^2 d\check{\sigma} d\tau$$

and:

$$\int \check{\sigma} \partial_{\check{\sigma}} |\check{p}\check{\psi}|^2 d\check{\sigma} d\tau = - \int |\check{p}\check{\psi}|^2 d\check{\sigma} d\tau.$$

Gathering the estimates, we obtain the lower bound:

$$\hat{Q}_{h,\text{app}}(\hat{\psi}) \geq \check{Q}_{h,\text{app}}(\check{\psi}) - Ch^{3/2+1/4} \|\check{\psi}\|^2.$$

## 5.2 A weighted Fourier transform: toward a model operator

We now define the unitary transform which diagonalizes the self-adjoint operator  $\check{p}^{-1}D_{\check{\sigma}}\check{p}$  (for completeness, one should extend  $\check{p}$  by 1 away from a neighborhood of 0). As we will see, with the coordinate  $\check{\sigma}$ , this transform admits a nice expression.

**Weighted Fourier transform** Let us now introduce the weighted Fourier transform  $\mathcal{F}_{\check{p}}$ :

$$(\mathcal{F}_{\check{p}}\psi)(\lambda) = \int_{\mathbb{R}} e^{-i\lambda\check{\sigma}} \psi(\check{\sigma}) \check{p}(\check{\sigma}) d\check{\sigma} = \mathcal{F}(\check{p}\psi).$$

We observe that  $\mathcal{F}_{\check{p}} : L^2(\mathbb{R}, \check{p}^2 d\check{\sigma}) \rightarrow L^2(\mathbb{R}, d\lambda)$  is unitary. Standard computations provide:

$$\mathcal{F}_{\check{p}}((\check{p}^{-1}D_{\check{\sigma}}\check{p})\psi) = \lambda \mathcal{F}_{\check{p}}(\psi)$$

and:

$$\mathcal{F}_{\check{p}}(\check{\sigma}\psi) = -D_{\lambda}\mathcal{F}_{\check{p}}(\psi).$$

**Proposition 5.5** *There exist  $h_0 > 0$  and  $C > 0$  such that for  $h \in (0, h_0)$  and all  $\check{\psi} \in \mathfrak{E}_N(h)$ :*

$$\begin{aligned} \check{Q}_{h,\text{app}}(\check{\psi}) &\geq \int (1 + a_2\tau)(1 - \tau k_0) |hD_{\tau}\check{\phi}|^2 d\lambda d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2)\check{\phi}|^2 d\lambda d\tau \\ &+ h\alpha\Theta_0 \int |D_{\lambda}\check{\phi}|^2 d\lambda d\tau - Ch^{3/2+1/4} \|\check{\psi}\|^2, \end{aligned}$$

where:

$$\check{\phi} = e^{-i\frac{b_1}{2h}\left(-h\lambda\frac{\tau^2}{2} + \frac{\tau^3}{3} - \xi_0 h^{1/2}\frac{\tau^2}{2} + \frac{b_2}{8}\tau^4\right)} \mathcal{F}_{\check{p}}(\check{\psi}).$$

*Proof:* We have:

$$\begin{aligned} \check{Q}_{h,\text{app}}(\check{\psi}) &= \int (1 + a_2\tau)(1 - \tau k_0) |h\partial_\tau \check{\varphi}|^2 d\lambda d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \check{\varphi}|^2 d\lambda d\tau \\ &+ h\alpha\Theta_0 \int |D_\lambda \check{\varphi}|^2 d\lambda d\tau, \end{aligned}$$

where

$$\check{\varphi} = \mathcal{F}_{\check{p}}(\check{\psi}).$$

With the normal estimates, we can write:

$$\begin{aligned} &\int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau) \check{\varphi}|^2 d\lambda d\tau \\ &\geq \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2) \check{\varphi}|^2 d\lambda d\tau \\ &- b_1 \Re \left( \int (1 + a_2\tau)(1 - \tau k_0)^{-1} (-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2) \check{\varphi} \overline{\tau h D_\tau \check{\varphi}} d\lambda d\tau \right) \\ &\geq \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2) \check{\varphi}|^2 d\lambda d\tau \\ &- b_1 \Re \left( \int (-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2) \check{\varphi} \overline{\tau h D_\tau \check{\varphi}} d\lambda d\tau \right) - Ch^{3/2+1/4} \|\check{\psi}\|^2. \end{aligned}$$

Completing a square and using the normal Agmon estimates to control the additional terms, we get:

$$\begin{aligned} &\check{Q}_{h,\text{app}}(\check{\psi}) \\ &\geq \int (1 + a_2\tau)(1 - \tau k_0) \left| \left( hD_\tau - \frac{b_1}{2}\tau \left( -h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 \right) \right) \check{\varphi} \right|^2 d\lambda d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2) \check{\varphi}|^2 d\lambda d\tau \\ &+ h\alpha\Theta_0 \int |D_\lambda \check{\varphi}|^2 d\lambda d\tau - Ch^{3/2+1/4} \|\check{\psi}\|^2. \end{aligned}$$

We now change the gauge by letting:

$$\check{\varphi} = e^{i\frac{b_1}{2h}\left(-h\lambda\frac{\tau^2}{2} + \frac{\tau^3}{3} - \xi_0 h^{1/2}\frac{\tau^2}{2} + \frac{b_2}{8}\tau^4\right)} \check{\phi}.$$



We deduce:

$$\begin{aligned}\check{Q}_{h,\text{app}}(\check{\psi}) &\geq \int (1 + a_2\tau)(1 - \tau k_0) |hD_\tau \check{\phi}|^2 d\lambda d\tau \\ &+ \int (1 + a_2\tau)(1 - \tau k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2) \check{\phi}|^2 d\lambda d\tau \\ &+ h\alpha\Theta_0 \int \left| D_\lambda \left( e^{-i\frac{\lambda b_1 \tau^2}{4}} \check{\phi} \right) \right|^2 d\lambda d\tau - Ch^{3/2+1/4} \|\check{\psi}\|^2.\end{aligned}$$

Finally, we write:

$$\begin{aligned}\int \left| D_\lambda \left( e^{-i\frac{\lambda b_1 \tau^2}{4}} \check{\phi} \right) \right|^2 d\lambda d\tau &= \int \left| D_\lambda \check{\phi} - \frac{b_1}{4} \tau^2 \check{\phi} \right|^2 d\lambda d\tau \\ &\geq \int |D_\lambda \check{\phi}|^2 d\lambda d\tau - C\|\tau^2 \check{\phi}\| \|D_\lambda \check{\phi}\| \geq \int |D_\lambda \check{\phi}|^2 d\lambda d\tau - C\|\tau^2 \check{\psi}\| \|D_\lambda \check{\phi}\|.\end{aligned}$$

In addition, we notice that:

$$\|D_\lambda \check{\phi}\| \leq C (\|\check{\sigma}\check{\psi}\| + \|\tau^2 \check{\psi}\|) \leq Ch^{1/4} \|\check{\psi}\|.$$

□

In order to get a good model operator, we shall add a cutoff function with respect to  $\tau$ . Let  $\eta \in (0, \frac{1}{100})$ . Let  $\chi$  a cutoff function such that:

$$\chi(t) = 1, \text{ for } |t| \leq 1, \quad 0 \leq \chi \leq 1 \text{ and } \text{supp} \chi \subset [-2, 2].$$

We define

$$l(x) = x\chi(h^\eta x).$$

Applying the normal Agmon estimates, we have:

**Proposition 5.6** *There exist  $h_0 > 0$  and  $C > 0$  such that for  $h \in (0, h_0)$  and all  $\check{\psi} \in \mathfrak{E}_N(h)$ :*

$$\begin{aligned}\check{Q}_{h,\text{app}}(\check{\psi}) &\geq \int (1 + a_2 h^{1/2} l(h^{-1/2} \tau))(1 - h^{1/2} l(h^{-1/2} \tau) k_0) |hD_\tau \check{\phi}|^2 d\lambda d\tau \\ &+ \int (1 + a_2 h^{1/2} l(h^{-1/2} \tau))(1 - h^{1/2} l(h^{-1/2} \tau) k_0)^{-1} |(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} h l(h^{-1/2} \tau)^2) \check{\phi}|^2 d\lambda d\tau \\ &+ h\alpha\Theta_0 \int |D_\lambda \check{\phi}|^2 d\lambda d\tau - Ch^{3/2+1/4} \|\check{\psi}\|^2,\end{aligned}$$

where:

$$\check{\phi} = e^{-i\frac{b_1}{2h} \left( -h\lambda \frac{\tau^2}{2} + \frac{\tau^3}{3} - \xi_0 h^{1/2} \frac{\tau^2}{2} + \frac{b_2}{8} \tau^4 \right)} \mathcal{F}_{\check{p}}(\check{\psi}).$$

**Remark 5.7** In particular we have reduced the analysis to an electric Laplacian (with curvature terms) which has essentially the Born-Oppenheimer form (see our recent work [6] where a similar and simpler model appears). To see this more precisely, let

us adopt a heuristical point of view. If we forget the different terms due to curvature, the operator which appears is in the form:

$$h\alpha\Theta_0 D_\lambda^2 + h^2 D_\tau^2 + (-h\lambda + \tau - \xi_0 h^{1/2})^2.$$

After the rescaling  $\lambda = h^{-1/4}\tilde{\lambda}$ ,  $\tau = h^{1/2}x$ , we get:

$$h \left( h^{1/2} \alpha \Theta_0 D_{\tilde{\lambda}}^2 + D_x^2 + (-h^{1/4}\tilde{\lambda} - x - \xi_0)^2 \right).$$

Therefore we are led to analyze a problem which is semiclassical with respect to just one variable. At some point (that we will justify at the end of this section), we can reduce the study to:

$$h \left( h^{1/2} \alpha \Theta_0 D_{\tilde{\lambda}}^2 + \mu(\xi_0 + h^{1/4}\tilde{\lambda}) \right)$$

and then (Taylor expansion):

$$h \left( h^{1/2} \alpha \Theta_0 D_{\tilde{\lambda}}^2 + \Theta_0 + \frac{\mu''(\xi_0)}{2} h^{1/2} \tilde{\lambda}^2 \right).$$

Finally we recognize the harmonic oscillator whose spectrum is well-known.

### 5.3 A simpler model in the Born-Oppenheimer spirit

We introduce the rescaled quadratic form:

$$\begin{aligned} Q_{\eta,h}(\varphi) &= \int (1 + a_2 h^{1/2} l(x)) (1 - l(x) k_0 h^{1/2}) |\partial_x \varphi|^2 d\lambda dx \\ &+ \int (1 + a_2 l(x) h^{1/2}) (1 - l(x) k_0 h^{1/2})^{-1} |(x - \xi_0 + h^{1/2} \lambda + \frac{\hat{b}_2}{2} l(x)^2 h^{1/2}) \varphi|^2 d\lambda dx \\ &+ \alpha \Theta_0 \int |D_\lambda \varphi|^2 d\lambda dx, \end{aligned}$$

We recall that  $\hat{b}_2 = b_2 - k_0$ . We will denote by  $H_{\eta,h}$  its corresponding Friedrichs extension. We will denote by  $\nu_n(Q_{\eta,h})$  the sequence of its Rayleigh quotients. For each  $\lambda$ , we will need to consider the following quadratic form:

$$\begin{aligned} q_{\lambda,\eta,h}(\varphi) &= \int (1 + a_2 h^{1/2} l(x)) (1 - l(x) k_0 h^{1/2}) |\partial_x \varphi|^2 dx \\ &+ \int (1 + a_2 l(x) h^{1/2}) (1 - l(x) k_0 h^{1/2})^{-1} |(x - \xi_0 + h^{1/2} \lambda + \frac{\hat{b}_2}{2} l(x)^2 h^{1/2}) \varphi|^2 dx, \end{aligned}$$

whose domain is  $B^1(\mathbb{R}^+)$ . We denote by  $\nu_j(q_{\lambda,\eta,h})$  the increasing sequence of the eigenvalues of the associated operator. The main proposition of this subsection is the following:

**Proposition 5.8** *For all  $n \geq 1$ , there exist  $h_0 > 0$  and  $C > 0$  s. t., for  $h \in (0, h_0)$ :*

$$\nu_n(Q_{\eta,h}) \geq \Theta_0 + \left( C(k_0, a_2, b_2) + (2n - 1) \sqrt{\frac{\alpha \mu''(\xi_0) \Theta_0}{2}} \right) h^{1/2} - C h^{1/2+1/8}.$$

Jointly with Propositions 5.6, 5.3, Inequality (5.1), the min-max principle, we first deduce the size of the spectral gap between the lowest eigenvalues of  $P_{h,\mathbf{A}}$ . Then, with Theorem 2.1, we deduce Theorem 1.3.

### 5.3.1 Elementary properties of the spectrum

This subsection is devoted to basic properties of the spectrum of  $Q_{\eta,h}$ . The following proposition provides a lower bound for  $\nu_1(q_{\lambda,\eta,h})$ .

**Proposition 5.9** *There exist positive constants  $C, c_0, M$  and  $h_0$  s.t. if  $h \in (0, h_0)$ , then:*

1. *If  $|\lambda| \geq Mh^{-1/4-\eta}$ , then:*

$$\nu_1(q_{\lambda,\eta,h}) \geq \Theta_0 + c_0 \min(1, \lambda^2 h).$$

2. *If  $|\lambda| \leq Mh^{-1/4-\eta}$ , then:*

$$\nu_1(q_{\lambda,\eta,h}) \geq \Theta_0 + C(k_0, a_2, b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h - Ch^{3/4-3\eta},$$

where  $C(k_0, a_2, b_2)$  is given in Theorem 1.3.

*Proof:* The proof is left to the reader as an adaptation of [14, Proposition 5.2.1].  $\square$

Let us now prove a lower bound for the essential spectrum of  $H_{\eta,h}$ .

**Proposition 5.10** *There exist  $h_0 > 0$  and  $\tilde{c}_0 > 0$  such that, if  $h \in (0, h_0)$ , then:*

$$\inf \sigma_{\text{ess}}(Q_{\eta,h}) \geq \Theta_0 + \tilde{c}_0$$

*Proof:* Let  $\phi \in \text{Dom}(Q_{\eta,h})$  such that  $\text{supp}(\phi) \subset \mathbb{R}_+^2 \setminus [-\tilde{R}, \tilde{R}]^2$ . Let us use a partition of unity:  $\chi_{1,R}^2 + \chi_{2,R}^2 = 1$  such that  $\chi_{1,R}(x) = \chi_1(R^{-1}x)$  and where  $\chi_1$  is a smooth cutoff function being 1 near 0. We have:

$$Q_{\eta,h}(\phi) \geq Q_{\eta,h}(\chi_{1,R}\phi) + Q_{\eta,h}(\chi_{2,R}\phi) - CR^{-2}\|\phi\|^2.$$

For  $R \geq 2h^{-\eta}$ , we have (the metrics becomes flat and we can compare with a problem in  $\mathbb{R}^2$ ):

$$Q_{\eta,h}(\chi_{2,R}\phi) \geq \|\chi_{2,R}\phi\|^2.$$

We have:

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h})|\chi_{1,R}\phi|^2 + \alpha\Theta_0|D_\lambda(\chi_{1,R}\phi)|^2 dx d\lambda$$

Taking  $h \in (0, h_0)$  (where  $h_0$  is given by Proposition 5.9) and  $\tilde{R} \geq h^{-1/2}$ , we infer:

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}_+^2} (\Theta_0 + c_0)|\chi_{1,R}\phi|^2 dx d\lambda.$$

This implies that:

$$Q_{\eta,h}(\phi) \geq (\min(1, \Theta_0 + c_0) - Ch^{2\eta})\|\phi\|^2.$$

The conclusion follows from a Persson's lemma-like argument (see [24] and also [14, Appendix B.3]).  $\square$

The following proposition provides an upper bound for the lowest eigenvalues of  $H_{\eta,h}$ .

**Proposition 5.11** *For all  $M \geq 1$ , there exist  $h_0 > 0$ ,  $C > 0$  s. t. for all  $1 \leq n \leq M$ :*

$$\nu_n(Q_{\eta,h}) \leq h^{-1}\lambda_n(h) + O(h^\infty).$$

*Proof:* This is a consequence of (5.1) joint with the lower bounds of Propositions 5.3 and 5.6 and the min-max principle (see for instance [28]).  $\square$

**Remark 5.12** For  $h$  small enough, we deduce that there is at least  $M$  eigenvalues below  $\Theta_0 + \tilde{c}_0$ . Let us consider the  $M$  first eigenvalues  $\nu_n(Q_{\eta,h})$  below  $\Theta_0 + \tilde{c}_0$ . With Theorem 2.1, we deduce that, for all  $M \geq 1$ , there exist  $h_0 > 0$  and  $C(M) > 0$  such that, for  $1 \leq n \leq M$ :

$$0 \leq \nu_n(Q_{\eta,h}) - \Theta_0 \leq C(M)h^{1/2}.$$

For  $1 \leq n \leq M$ , let us consider a normalized eigenfunction  $f_{n,\eta,h}$  associated to  $\nu_n(Q_{\eta,h})$  so that  $f_{n,\eta,h}$  and  $f_{m,\eta,h}$  are orthogonal if  $n \neq m$ . Let us introduce:

$$\mathfrak{F}_M(h) = \text{span}_{1 \leq j \leq M}(f_{j,\eta,h}).$$

### 5.3.2 Agmon estimates

First, let us state Agmon estimates with respect to  $x$ .

**Proposition 5.13** *There exist  $h_0 > 0$ ,  $\varepsilon_0 > 0$ ,  $C > 0$  such that, for all  $f \in \mathfrak{F}_M(h)$ :*

$$\int_{\mathbb{R}_+^2} e^{\varepsilon_0 x} |f|^2 dx d\lambda \leq C \|f\|^2.$$

*Proof:* Let us use a partition of unity:  $\chi_{1,R}^2 + \chi_{2,R}^2 = 1$ , with  $R \geq h^{-\eta}$ . We take  $\Phi = \varepsilon_0 \chi\left(\frac{x}{r}\right) |x|$ . This IMS formula implies (with  $f = f_{n,\eta,h}$ ):

$$Q_{\eta,h}(\chi_{1,R} e^\Phi f) + Q_{\eta,h}(\chi_{2,R} e^\Phi f) - C\varepsilon_0^2 \|e^\Phi f\|^2 - \nu_n(Q_{\eta,h}) \|e^\Phi f\|^2 \leq 0.$$

We recall that:

$$Q_{\eta,h}(\chi_{2,R} e^\Phi f) \geq \|\chi_{2,R} e^\Phi f\|^2$$

and that:

$$Q_{\eta,h}(\chi_{1,R} e^\Phi f) \geq \int \nu_1(q_{\lambda,\eta,h}) |\chi_{1,R} e^\Phi f|^2 dx d\lambda.$$

On the one hand, we have:

$$\begin{aligned} Q_{\eta,h}(\chi_{2,R}e^\Phi f) - C\varepsilon_0^2\|\chi_{2,R}e^\Phi f\|^2 - (\Theta_0 + Ch^{1/2})\|\chi_{2,R}e^\Phi f\|^2 \\ \geq (1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2})\|\chi_{2,R}e^\Phi f\|^2. \end{aligned}$$

On the other hand, we get:

$$\begin{aligned} Q_{\eta,h}(\chi_{1,R}e^\Phi f) - C\varepsilon_0^2\|\chi_{1,R}e^\Phi f\|^2 - (\Theta_0 + Ch^{1/2})\|\chi_{1,R}e^\Phi f\|^2 \\ \geq \int (\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2})|\chi_{1,R}e^\Phi f|^2 dx d\lambda. \end{aligned}$$

When  $|\lambda| \geq Mh^{-1/4-\eta}$ , we have:

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - Ch^{1/2}.$$

When  $|\lambda| \leq Mh^{-1/4}$ , we have:

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - \tilde{C}h^{1/2}.$$

If  $h$  and  $\varepsilon_0$  are small enough, we deduce that:

$$(1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2})\|\chi_{2,R}e^\Phi f\|^2 \leq C\|\chi_{1,R}e^\Phi f\|^2$$

so that:

$$\|\chi_{2,R}e^\Phi f\|^2 \leq \tilde{C}\|f\|^2 \text{ and } \|e^\Phi f\|^2 \leq \hat{C}\|f\|^2,$$

where  $\tilde{C}$  and  $\hat{C}$  are independent from  $r$ . It remains to make  $r \rightarrow +\infty$  and apply the Fatou lemma. Finally, this is easy to extend the inequality to  $f \in \mathfrak{F}_M(h)$ .  $\square$

Then, we will need Agmon estimates with respect to  $\lambda$ :

**Proposition 5.14** *There exist  $h_0 > 0$ ,  $C > 0$  such that, for all  $f \in \mathfrak{F}_M(h)$ :*

$$\int_{\mathbb{R}_+^2} e^{2h^{1/4}|\lambda|} |f|^2 dx d\lambda \leq C\|f\|^2 \quad (5.4)$$

and:

$$\int_{\mathbb{R}_+^2} e^{2h^{1/4}|\lambda|} |D_\lambda f|^2 dx d\lambda \leq Ch^{1/2}\|f\|^2 \quad (5.5)$$

**Remark 5.15** Heuristically, those estimates with respect to  $\lambda$  correspond to the phase space localization of [13, Section 5].

*Proof:* We take  $f = f_{j,\eta,h}$  and we use the IMS formula (with  $\Phi = h^{1/4}\chi(r^{-1}|\lambda|)|\lambda|$ ) to get:

$$Q_{\eta,h}(e^\Phi f) \leq \nu_j(Q_{\eta,h})\|e^\Phi f\|^2 + C\|\nabla\Phi e^\Phi f\|^2 \leq (\Theta_0 + C(M)h^{1/2} + Ch^{1/2})\|e^\Phi f\|^2.$$

We recall that:

$$\begin{aligned} Q_{\eta,h}(e^\Phi f) &\geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h}) |e^\Phi f|^2 + \alpha \Theta_0 |D_\lambda(e^\Phi f)|^2 dx d\lambda \\ &\geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h}) |e^\Phi f|^2 dx d\lambda. \end{aligned}$$

We have, for all  $D > 0$ :

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h}) |e^\Phi f|^2 dx d\lambda \\ &= \int_{|\lambda| \leq Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h}) |e^\Phi f|^2 dx d\lambda + \int_{|\lambda| \geq Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h}) |e^\Phi f|^2 dx d\lambda. \end{aligned}$$

Moreover, we get:

$$\begin{aligned} &\int_{|\lambda| \geq Mh^{-1/4-\eta}} \nu_1(q_{\lambda,\eta,h}) |e^\Phi f|^2 dx d\lambda \\ &\geq \int_{|\lambda| \geq Mh^{-1/4-\eta}} (\Theta_0 + c_0 \min(1, h\lambda^2)) |e^\Phi f|^2 dx d\lambda \end{aligned}$$

and:

$$\begin{aligned} &\int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \nu_1(q_{\lambda,\eta,h}) |e^\Phi f|^2 dx d\lambda \\ &\geq \int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 dx d\lambda. \end{aligned}$$

This leads to:

$$\begin{aligned} &\int_{|\lambda| \geq Dh^{-1/4}} (c_1 \min(1, h\lambda^2) - \tilde{C} h^{1/2} - C\alpha^2 h^{1/2}) |e^\Phi f|^2 dx d\lambda \\ &\leq \tilde{C} h^{1/2} \int_{|\lambda| \leq Dh^{-1/4}} |f|^2 d\lambda dx. \end{aligned}$$

It remains to take  $D$  large enough and we get (5.4). Then, we have:

$$\int_{\mathbb{R}_+^2} (\nu_1(q_{\lambda,\eta,h}) - \Theta_0) |e^\Phi f|^2 + \alpha \Theta_0 |D_\lambda(e^\Phi f)|^2 dx d\lambda \leq Ch^{1/2} \|f\|^2.$$

But, we notice that:

$$\begin{aligned} &\int_{\mathbb{R}_+^2} (\nu_1(q_{\lambda,\eta,h}) - \Theta_0) |e^\Phi f|^2 dx d\lambda \\ &\geq \int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left( C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 dx d\lambda \\ &+ \int_{|\lambda| \leq Dh^{-1/4}} \left( C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 d\lambda dx. \end{aligned}$$

Taking  $D$  larger, we get:

$$\int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left( \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{1/2} - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 dx d\lambda \geq 0.$$

Moreover, we have:

$$\left| \int_{|\lambda| \leq Dh^{-1/4}} \left( C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - Ch^{3/4-3\eta} \right) |e^\Phi f|^2 d\lambda dx \right| \leq Ch^{1/2} \|f\|^2.$$

□

### 5.3.3 Approximations of eigenvectors by tensor products

Let us define the quadratic form  $q_0$  with domain  $B^1(\mathbb{R}_+) \otimes L^2(\mathbb{R})$ :

$$q_0(\varphi) = Q_0(\varphi) - \Theta_0 \|\varphi\|^2 = \int_{\mathbb{R}_+^2} |\partial_x \varphi|^2 + |(x - \xi_0)\varphi|^2 - \Theta_0 |\varphi|^2 dx d\lambda.$$

The Friedrichs extension of  $q_0$  is the operator  $H_{\xi_0} \otimes \text{Id}_{L^2(\mathbb{R})}$ . We also define the Feshbach-Grushin projection on the kernel of  $H_{\xi_0} \otimes \text{Id}_{L^2(\mathbb{R})}$ :

$$\Pi_0 \varphi = \langle \varphi, u_{\xi_0} \rangle_x u_{\xi_0}(x).$$

The next proposition states an approximation result for the elements of  $\mathfrak{F}_M(h)$  (which behave as tensor products):

**Proposition 5.16** *For all  $M \geq 1$ , there exist  $h_0 > 0$  and  $C > 0$  such that, we have, for all  $f \in \mathfrak{F}_M(h)$ :*

$$\|f - \Pi_0 f\|_{L^2} + \|\partial_x(f - \Pi_0 f)\|_{L^2} + \|x(f - \Pi_0 f)\|_{L^2} \leq Ch^{1/8} \|f\|, \quad (5.6)$$

$$\|(\lambda f - \Pi_0 \lambda f)\|_{L^2} + \|\partial_x(\lambda f - \Pi_0 \lambda f)\|_{L^2} + \|x(\lambda f - \Pi_0 \lambda f)\|_{L^2} \leq Ch^{-1/8} \|f\|, \quad (5.7)$$

$$\|(\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} + \|\partial_x(\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} + \|x(\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} \leq Ch^{3/8} \|f\|. \quad (5.8)$$

In particular,  $\Pi_0$  is an isomorphism from  $\mathfrak{F}_M(h)$  onto its range.

*Proof:* We take  $f = f_{j,\eta,h}$ . By definition, we have:

$$H_{\eta,h} f = \nu_j(Q_{\eta,h}) f. \quad (5.9)$$

**Approximation of  $f$**  We deduce:

$$Q_{\eta,h}(f) = \nu_j(Q_{\eta,h}) \|f\|^2 \leq (\Theta_0 + Ch^{1/2}) \|f\|^2.$$

We have:

$$Q_{\eta,h}(f) \geq (1 - Ch^{1/2-\eta}) \int_{\mathbb{R}_+^2} |\partial_x f|^2 + |(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{\hat{b}_2}{2}l(x)^2)f|^2 dx d\lambda.$$

Moreover, we get (using the estimates of Agmon), for all  $\varepsilon \in (0, 1)$ :

$$\begin{aligned} & \int_{\mathbb{R}_+^2} |\partial_x f|^2 + |(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{b_2}{2}l(x)^2)f|^2 dx d\lambda \\ & \geq (1 - \varepsilon)Q_0(f) - C\varepsilon^{-1}h^{1/2}\|f\|^2. \end{aligned}$$

Taking  $\varepsilon = h^{1/4}$ , we deduce:

$$q_0(f) \leq Ch^{1/4}\|f\|^2.$$

We deduce (5.6).

**Approximation of  $\lambda f$**  We multiply (5.9) by  $\lambda$  and take the scalar product with  $\lambda f$ :

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + |\langle [H_{\eta,h}, \lambda]f, \lambda f \rangle|.$$

Thus, it follows:

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + \alpha\Theta_0|\langle D_\lambda f, \lambda f \rangle| \leq \Theta_0\|\lambda f\|^2 + C\|f\|^2.$$

We get:

$$Q_{\eta,h}(\lambda f) \geq (1 - Ch^{1/2-\eta})((1 - \varepsilon)Q_0(\lambda f) - C\varepsilon^{-1}\|f\|^2).$$

We take  $\varepsilon = h^{1/4}$  to deduce:

$$q_0(\lambda f) \leq Ch^{-1/4}\|f\|^2.$$

We infer (5.7).

**Approximation of  $D_\lambda f$**  We take the derivative of (5.9) with respect to  $\lambda$  and take the scalar product with  $\partial_\lambda f$ :

$$Q_{\eta,h}(\partial_\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\partial_\lambda f\|^2 + |\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f \rangle|.$$

The estimates of Agmon give:

$$|\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f \rangle| \leq Ch^{3/4}\|f\|^2.$$

We have:

$$Q_{\eta,h}(\partial_\lambda f) \geq (1 - Ch^{1/2-\eta})((1 - \varepsilon)Q_0(\partial_\lambda f) - C\varepsilon^{-1}h\|f\|^2).$$

We take  $\varepsilon = h^{1/4}$  and we deduce:

$$q_0(\partial_\lambda f) \leq Ch^{3/4}\|f\|^2.$$

We infer (5.8).

□



### 5.3.4 Conclusion: proof of Proposition 5.8

For all  $f \in \mathfrak{F}_M(h)$ , we have the lower bound:

$$\begin{aligned} Q_{\eta,h}(f) &\geq \int_{\mathbb{R}_+^2} \nu_1(q_{\lambda,\eta,h}) |f|^2 + \alpha \Theta_0 |D_\lambda f|^2 dx d\lambda \\ &\geq \int_{\mathbb{R}_+^2} \left( \nu_1(q_{\lambda,\eta,h}) - \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\quad + \int_{\mathbb{R}_+^2} \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda f|^2 dx d\lambda \end{aligned}$$

We now estimate:

$$\begin{aligned} &\int_{\mathbb{R}_+^2} \left( \nu_1(q_{\lambda,\eta,h}) - \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &= \int_{|\lambda| \geq Mh^{-1/4-\eta}} \left( \nu_1(q_{\lambda,\eta,h}) - \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\quad + \int_{|\lambda| \leq Mh^{-1/4-\eta}} \left( \nu_1(q_{\lambda,\eta,h}) - \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \end{aligned}$$

Moreover, we get:

$$\begin{aligned} &\int_{|\lambda| \geq Mh^{-1/4-\eta}} \left( \nu_1(q_{\lambda,\eta,h}) - \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\geq \int_{|\lambda| \geq Mh^{-1/4-\eta}} - \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |f|^2 dx d\lambda = O(h^\infty) \|f\|^2, \end{aligned}$$

where the last estimate is a consequence of the estimates of Agmon. Then, we get:

$$\begin{aligned} &\int_{|\lambda| \leq Mh^{-1/4-\eta}} \left( \nu_1(q_{\lambda,\eta,h}) - \left( \Theta_0 + C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) \right) |f|^2 dx d\lambda \\ &\geq -Ch^{3/4-3\eta} \|f\|^2. \end{aligned}$$

We deduce:

$$\begin{aligned} Q_{\eta,h}(f) &\geq \int_{\mathbb{R}_+^2} \left( C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda f|^2 dx d\lambda \\ &\quad + \Theta_0 \|f\|^2 - Ch^{3/4-3\eta} \|f\|^2. \end{aligned}$$

We now use Proposition 5.16 to get:

$$\begin{aligned} Q_{\eta,h}(f) &\geq \int_{\mathbb{R}_+^2} \left( C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |\Pi_0 f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda \Pi_0 f|^2 dx d\lambda \\ &\quad + \Theta_0 \|f\|^2 - Ch^{1/2+1/8} \|\Pi_0 f\|^2. \end{aligned}$$

But, we notice that, for all  $f \in \mathfrak{F}_M(h)$ :

$$Q_{\eta,h}(f) \leq \nu_M(Q_{\eta,h}) \|f\|^2$$

and thus:

$$\begin{aligned}
& \int_{\mathbb{R}_+^2} \left( C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h \right) |\Pi_0 f|^2 dx d\lambda + \alpha \Theta_0 |D_\lambda \Pi_0 f|^2 dx d\lambda \\
& \leq (\nu_M(Q_{\eta, h}) - \Theta_0) \|f\|^2 + C h^{1/2+1/8} \|\Pi_0 f\|^2 \\
& \leq (\nu_M(Q_{\eta, h}) - \Theta_0) \|\Pi_0 f\|^2 + \tilde{C} h^{1/2+1/8} \|\Pi_0 f\|^2.
\end{aligned}$$

The conclusion follows from the min-max principle.

## References

- [1] S. AGMON. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, volume 29 of *Mathematical Notes*. Princeton University Press, Princeton, NJ 1982.
- [2] S. AGMON. Bounds on exponential decay of eigenfunctions of Schrödinger operators. In *Schrödinger operators (Como, 1984)*, volume 1159 of *Lecture Notes in Math.*, pages 1–38. Springer, Berlin 1985.
- [3] S. ALAMA, L. BRONSARD, B. GALVÃO-SOUSA. Thin film limits for Ginzburg-Landau with strong applied magnetic fields. *SIAM J. Math. Anal.* **42**(1) (2010) 97–124.
- [4] P. BAUMAN, D. PHILLIPS, Q. TANG. Stable nucleation for the Ginzburg-Landau system with an applied magnetic field. *Arch. Rational Mech. Anal.* **142**(1) (1998) 1–43.
- [5] A. BERNOFF, P. STERNBERG. Onset of superconductivity in decreasing fields for general domains. *J. Math. Phys.* **39**(3) (1998) 1272–1284.
- [6] V. BONNAILLIE, M. DAUGE, N. POPOFF, N. RAYMOND. Discrete spectrum of a model schrödinger operator on the half-plane with neumann conditions. *Z. Ange. Math. Phys.* (2011).
- [7] S. J. CHAPMAN, Q. DU, M. D. GUNZBURGER. On the Lawrence-Doniach and anisotropic Ginzburg-Landau models for layered superconductors. *SIAM J. Appl. Math.* **55**(1) (1995) 156–174.
- [8] H. L. CYCON, R. G. FROESE, W. KIRSCH, B. SIMON. *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer-Verlag, Berlin, study edition 1987.
- [9] M. DAUGE, B. HELFFER. Eigenvalues variation. I. Neumann problem for Sturm-Liouville operators. *J. Differential Equations* **104**(2) (1993) 243–262.
- [10] M. DEL PINO, P. L. FELMER, P. STERNBERG. Boundary concentration for eigenvalue problems related to the onset of superconductivity. *Comm. Math. Phys.* **210**(2) (2000) 413–446.
- [11] M. M. DORIA, S. C. B. DE ANDRADE. Virial theorem for the anisotropic Ginzburg-Landau theory. *Phys. Rev. B* **53** (Feb 1996) 3440–3454.

- [12] J. V. EGOROV. Canonical transformations and pseudodifferential operators. *Trudy Moskov. Mat. Obšč.* **24** (1971) 3–28.
- [13] S. FOURNAIS, B. HELFFER. Accurate eigenvalue asymptotics for the magnetic Neumann Laplacian. *Ann. Inst. Fourier (Grenoble)* **56**(1) (2006) 1–67.
- [14] S. FOURNAIS, B. HELFFER. *Spectral methods in surface superconductivity*. Progress in Nonlinear Differential Equations and their Applications, 77. Birkhäuser Boston Inc., Boston, MA 2010.
- [15] V. V. GRUŠIN. Hypoelliptic differential equations and pseudodifferential operators with operator-valued symbols. *Mat. Sb. (N.S.)* **88(130)** (1972) 504–521.
- [16] B. HELFFER. *Semi-classical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin 1988.
- [17] B. HELFFER, Y. A. KORDYUKOV. Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator: the case of discrete wells. In *Spectral theory and geometric analysis*, volume 535 of *Contemp. Math.*, pages 55–78. Amer. Math. Soc., Providence, RI 2011.
- [18] B. HELFFER, A. MORAME. Magnetic bottles in connection with superconductivity. *J. Funct. Anal.* **185**(2) (2001) 604–680.
- [19] B. HELFFER, A. MORAME. Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case). *Ann. Sci. École Norm. Sup. (4)* **37**(1) (2004) 105–170.
- [20] T. KATO. *Perturbation theory for linear operators*. Die Grundlehren der mathematischen Wissenschaften, Band 132. Springer-Verlag New York, Inc., New York 1966.
- [21] K. LU, X.-B. PAN. Eigenvalue problems of Ginzburg-Landau operator in bounded domains. *J. Math. Phys.* **40**(6) (1999) 2647–2670.
- [22] K. LU, X.-B. PAN. Surface nucleation of superconductivity in 3-dimensions. *J. Differential Equations* **168**(2) (2000) 386–452. Special issue in celebration of Jack K. Hale’s 70th birthday, Part 2 (Atlanta, GA/Lisbon, 1998).
- [23] A. MARTINEZ. Développements asymptotiques et effet tunnel dans l’approximation de Born-Oppenheimer. *Ann. Inst. H. Poincaré Phys. Théor.* **50**(3) (1989) 239–257.
- [24] A. PERSSON. Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. *Math. Scand.* **8** (1960) 143–153.
- [25] N. RAYMOND. Sharp asymptotics for the Neumann Laplacian with variable magnetic field: case of dimension 2. *Ann. Henri Poincaré* **10**(1) (2009) 95–122.
- [26] N. RAYMOND. On the semiclassical 3D Neumann Laplacian with variable magnetic field. *Asymptot. Anal.* **68**(1-2) (2010) 1–40.
- [27] N. RAYMOND. Semiclassical 3D Neumann Laplacian with variable magnetic field: a toy model. *To appear in Comm. in PDE* (2012).

- [28] M. REED, B. SIMON. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York 1978.